Representation of Concave Distortions and Applications

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Abstract

A family of concave distortion functions is a set of concave and increasing functions, mapping the unity interval onto itself. Distortion functions play an important role defining coherent risk measures. We prove that any family of distortion functions which fulfils a certain translation equation, can be represented by a distribution function. An application can be found in actuarial science: moment based premium principles are easy to understand but in general are not monotone and cannot be used to compare the riskiness of different insurance contracts with each other. Our representation theorem makes it possible to compare two insurance risks with each other consistent with a moment based premium principle by defining an appropriate coherent risk measure.

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Key-Words: representation of distortion functions; premium principle; coherent risk measure; WANG-transform; log-concavity

1 Introduction

Concave distortion functions play a very important role in insurance and financial mathematics. They are used to define coherent risk measures, as introduced axiomatically by Artzner et al. (1999). Risk measures are for example applied by insurances to compute the premium of an insurance contract or may describe a potential loss from a capital investment.

A concave distortion function widely used in actuarial science is the WANG-transform, defined by

$$\Psi^{\gamma}_{\text{WANG}}(u) = \Phi(\Phi^{-1}(u) + \gamma), \quad u \in [0, 1], \quad \gamma \geq 0, \tag{1}$$

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which involves the standard cumulative normal distribution Φ and its inverse, see Wang (2000).

In this article, generalizations of the WANG-transform play a special role: we will prove a representation theorem and show that a family of concave distortion functions (FCDF) satisfying a certain translation equation can be represented by a distribution function G.

It is well known that the coherent risk measure induced by the WANG-transform reduces to the standard deviation premium principle for normal distributed random variables. Our representation theorem helps to interpret general FCDF in a similar spirit.

An application of this theorem can be found in insurance science. Premium principles in actuarial science are used to determine the premium an insured has to pay to the insurance company in return for an insurance contract. For example the premium can be calculated by the expected loss of the insured object plus a multiple of the standard deviation of the loss. Such moment based premium principles are easy to understand but in general are not monotone and cannot be used to compare the riskiness of different insurance contracts with each other. Our representation theorem makes it possible to compare two insurance risks with each other consistent with a moment based premium principle by defining an appropriate coherent risk measure.

In particular, we answer the following question: if an insurance company insures risk X for a certain premium and the premium is computed using a classical moment based premium principle, what would be an adequate premium for another risk Z consistent with the premium of X? We are able to answer this question even if Z as infinite second moments. Consistency between the premium for X and for Z is measured using performance measures as axiomatically introduced by Cherny and Madan (2009).

In Section 2 we define a coherent risk measures via concave distortion functions. In Section 3 the translation equation for a family of concave distortion functions (FCDF) is defined. In Section 4 we present our main theorem, which provides a connection between FCDF and distribution functions. We discuss under which conditions a general FCDF can be reparameterized into a FCDF satisfying a translation equation and provide various examples. In Section 5 we construct a coherent risk measure which makes it possible to compare two insurance risks with each other consistent with a moment based premium principle.

2 Coherent Risk Measures

A coherent risk measure maps the set of bounded random variables to the real numbers fulfilling four axioms:

Definition 2.1. (Coherent risk measure). A map $\rho : L^\infty \to \mathbb{R}$ is called a *coherent risk measure* if it satisfies the following properties for all $X, Y \in L^\infty$:

R1: *Cash invariance:* $\rho(X + c) = \rho(X) + c$ for any $c \in \mathbb{R}$.

R2: *Monotonicity:* $X \le Y \Rightarrow \rho(X) \le \rho(Y)$.

R3: *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $0 \le \lambda \le 1$.

R4: *Positive homogeneity:* $\rho(\lambda X) = \lambda\rho(X)$ where $0 \le \lambda$.

Throughout this article, we assume that a random variable Y describes the loss of a financial position, not a net worth. Coherent risk measures can be seen as premium principles in insurance science.

Let Ψ be a concave, increasing function, mapping the unity interval onto itself such that $\Psi(0) = 0$ and $\Psi(1) = 1$. Ψ is called *distortion function.* According to Föllmer and Schied (2011, Theorem 4.70 and Theorem 4.93), see also Kusuoka (2001), a law invariant and comonotonic coherent risk measure can be defined for $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\rho_\Psi(Y) = \int_{-\infty}^{0} \left(\Psi\left(\mathbb{P}\left[Y > y\right]\right) - 1\right) dy + \int_{0}^{\infty} \Psi\left(\mathbb{P}\left[Y > y\right]\right) dy \tag{2}$$

$$= \Psi(0+)\operatorname{ess\,sup}\{Y\} + \int_{0}^{1} F_Y^{-1}(y) d\hat{\Psi}(y), \tag{3}$$

where we define the convex *dual* distortion by

$$\hat{\Psi}(u) = 1 - \Psi(1-u).$$

The value

$$\Psi(0+) := \lim_{\varepsilon\downarrow 0} \Psi(\varepsilon)$$

denotes the jump-size at $u = 0$ of the distortion function and

$$\operatorname{ess\,sup}\{Y\} := \inf\{m \in \mathbb{R} : m \ge Y,\, \mathbb{P} - \text{a.s.}\}$$

describes the essential supremum of Y. We say the risk measure ρ_Ψ is *induced by the distortion function* Ψ. If Ψ is equal to the identity, it holds $\rho_\Psi(Y) = E[Y]$.

Remark 2.2. If the distortion function Ψ is continuous and differentiable with bounded derivative, the functional ρ_Ψ is well defined on L^1, see Pichler (2013).

Remark 2.3. Some authors define a coherent risk measure via Equation (2), see Wang (2000, eq. (2)) and Tsanakas (2004, eq. (3)). Acerbi (2002) and Tsukahara (2009, eq. (1.1)) among others work with the convex dual distortion and use Equation (3) to define coherent risk measures. In contrast to actuarial science, in the financial literature, it is common to interpret a random variable as the net worth of a financial position. A coherent risk measure is then defined via a concave distortion function by $\rho_\Psi(-.)$, i.e. the sign is changed, see for example Artzner et al. (1999), Kusuoka (2001), Cherny and Madan (2009) and Föllmer and Schied (2011).

3 Family of Concave Distortion Functions

Often, one would like work with a parametric family of risk measures $(\rho_\gamma)_{\gamma\geq 0}$, where γ models the view of the risk manager: the greater γ, the more conservative the risk measure ρ_γ. For example Wang (1995) and Wang (2000) proposed the proportional hazard transform and the WANG-transform as distortion functions for insurance premium calculation of an insurance risk $X \geq 0$. The premium is computed according to Equation (2). Both distortions depend on a single parameter γ: the premium of a risk is thus a function of γ and varies continuously between the smallest and greatest reasonably premium: the expected value and maximal value of X. The insurance company may choose γ depending on many external circumstances and the risk-attitude of the company. Wang (2000) proposed that possible changes in court rulings or in the interest rate yield curve, moral hazards by insurance buyers and competition with other insurance companies, should be taken into consideration when choosing the parameter γ.

Another use of a family of risk measures is discussed in Cherny and Madan (2009), who proved that an acceptability index, which measures the performance of a future random cash flow, can be represented by an increasing family of coherent risk measures.

If the parametric family of risk measures is induced by distortion functions, we need to work with a family of concave distortion functions, which is defined as follows:

Definition 3.1. A *family of concave distortion functions* (FCDF) $(\Psi^\gamma)_{\gamma\geq 0}$ is a set of functions $\Psi^\gamma : [0,1] \to [0,1]$ that are monotonically increasing and concave for all $\gamma \geq 0$ and for which $\Psi^\gamma(0) = 0$ and $\Psi^\gamma(1) = 1$. Moreover the family is monotonically increasing and continuous at γ, i.e. it holds that for all $u \in [0,1]$: $\Psi^{\gamma_1}(u) \leq \Psi^{\gamma_2}(u)$ for $\gamma_1 \leq \gamma_2$ and the map $\gamma \mapsto \Psi^\gamma(u)$ is continuous for all $u \in [0,1]$.

We note that the map $u \mapsto \Psi^\gamma(u)$ is continuous on $(0,1]$ for all $\gamma \geq 0$ but might jump at zero, see Rockafellar (1970, Theorem 10.1). Let us additionally assume the following conditions:

[E] It holds $\Psi^0(u) = u$, for $u \in [0,1]$.

[W] It holds $\lim_{\gamma\to\infty} \Psi^\gamma(u) = 1$, for $u \in (0,1]$.

[T] It holds $\Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u)\right) = \Psi^{\gamma_1+\gamma_2}(u)$, for $\gamma_1, \gamma_2 \geq 0$ and $u \in [0,1]$.

The interpretation of Definition 3.1 is the following: the greater γ, the greater the distortion and the more conservative the risk measure induced by Ψ^γ. Conditions [E] and [W] are quite natural: it is usually desirable that for $\gamma = 0$ no distortion occurs, the risk measure induced by Ψ^0 should be equal to the expectation operator.

For $\gamma \to \infty$ the risk measure induced by Ψ^γ should converge to the worst-case risk measure, i.e. $\Psi^\gamma(u)$ should converge to 1 for $u > 0$, which is expressed in condition [W].

Condition [T] means distorting the probability u first at level γ_1 and then at level γ_2 is the same as distorting the probability once at level $\gamma_1 + \gamma_2$. This condition is called *translation equation* in functional equation theory, see Aczél (1966, Section 6.1.1.).

4 Duality between Distortion and Distribution Functions

In the following theorem, we present our main result: a relationship between distribution functions and FCDF. An application to insurance science can be found in Section 5.

Theorem 4.1. *Let* (Ψ^γ) *be a FCDF. Let* $u_0 \in (0,1)$. *The following two statements are equivalent.*

i) *The* FCDF (Ψ^γ) *satisfies conditions [E], [W] and [T].*

ii) *There exists a unique distribution function* G*, such that* $G(0) = u_0$ *and*

$$\Psi^\gamma(u) = G(G^{-1}(u) + \gamma), \quad \gamma \geq 0, \quad u \in (0,1). \tag{4}$$

Proof. The proof is devoted to the Appendix. ◻

Remark 4.2. For a given distribution function G a family of functions (Ψ^γ) defined by Equation (4) is a FCDF if $g = G'$ is log-concave, see Tsukahara (2009, p. 697). The function g is called *log-concave* if $\log(g)$ is concave. See Dharmadhikari and Joag-Dev (1988) for log-concavity and related topics.

The constant u_0 mentioned in the theorem can be chosen arbitrarily: if G induces Ψ^γ then also the shifted distribution $\tilde{G}(x) := G(x + \mu)$ for any $\mu \in \mathbb{R}$ induces Ψ^γ. Hence we could reformulate Theorem 4.1 and say that G is unique up to location translation. The distribution function G can be identified by

$$G(x) = \begin{cases} \Psi^x(u_0) & , x \geq 0 \\ \overline{\Psi}^{-x}(u_0) & , x < 0, \end{cases} \tag{5}$$

where $\overline{\Psi}^\gamma$ is the generalized inverse of the function $u \mapsto \Psi^\gamma(u)$, in particular for $\gamma \geq 0$

$$\begin{aligned} \overline{\Psi}^\gamma : [0,1] &\to [0,1] \\ p &\mapsto \inf\{u \in [0,1] : \Psi^\gamma(u) \geq p\}. \end{aligned}$$

Remark 4.3. Based on results from functional equation theory, see Aczél (1966, Section 6.1.), Tsukahara (2009) obtained a similar result, under the additional assumptions that the FCDF is continuous in the variable u and strictly increasing in the variable γ and that G is strictly increasing. Tsukahara works with the *convex dual* of the concave distortion function Ψ to define coherent risk measures, see Remark 2.3. Examples 4.4 - 4.7 provide various FCDF used in practise, which are not continuous at $u = 0$ or are not strictly increasing in the variable γ but can be represented by a distribution function. Some of those FCDF are applied in Section 5 to actuarial science and we develop a new FCDF using the gamma distribution, which includes the expected shortfall and the WANG-transform as special cases.

We provide four examples of FCDF satisfying conditions [E], [W] and [T]. The four distortions are also shown in Figure 1.

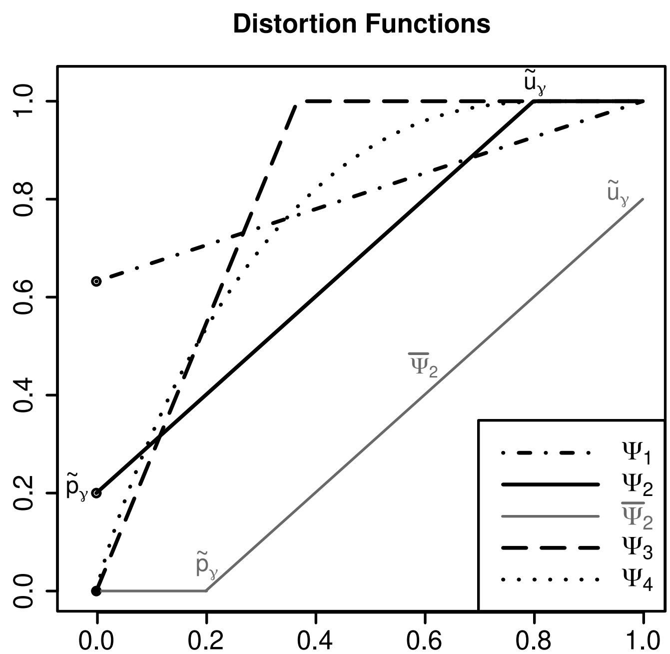

Figure 1: Distortion Functions from Examples 4.4 - 4.7. We set $\gamma = 1$. $\overline{\Psi}_2$ denotes the generalized inverse of Ψ_2. The jump-size of Ψ_2 at zero is defined by $\tilde{p}_\gamma$ and the point, where Ψ_2 first reaches one, is defined by $\tilde{u}_\gamma$.

Example 4.4. The following FCDF is not continuous at $u = 0$. Let

$$\Psi_1^\gamma(u) := \begin{cases} 0 & , u = 0 \\ 1 - (1-u)e^{-\gamma} & , u > 0, \end{cases}$$

The FCDF (Ψ_1^γ) is called "ess sup-*expectation convex combination*" by Bannör and Scherer (2014) because the coherent risk measure induced by (Ψ_1^γ) involves a convex combination of the essential supremum and the ordinary expectation. Bannör and Scherer (2014) applied this FCDF to calibrate a non-linear pricing model to quoted bid-ask prices. $(\Psi_1^\gamma)_{\gamma\geq 0}$ is induced by the exponential distribution function

$$G_1(x) = \begin{cases} 1 - e^{-x} & , x > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Example 4.5. Let

$$\Psi_2^\gamma(u) := \begin{cases} 0 & , u = 0 \\ \min\left(u + \frac{\gamma}{\lambda}, 1\right) & , u > 0. \end{cases}$$

This FCDF is induced by the uniform distribution function on $\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$ for any $\lambda > 0$.

Example 4.6. The FCDF corresponding to the expected shortfall at level $e^{-\gamma} \in (0, 1]$, see e.g. Föllmer and Schied (2011, Example 4.71), can be defined by

$$\Psi_3^\gamma(u) := \min(ue^\gamma, 1).$$

This FCDF is induced by the distribution $G_3(x) = \min(e^x, 1)$, $x \in \mathbb{R}$ and is increasing in the variable γ but not strictly increasing.

Let X be exponential distributed. It holds

$$\rho_{\Psi_3^\gamma}(X) = E[X](1 + \gamma),$$

i.e. the expected shortfall reduces to the expected value premium principle when applied to exponential risks.

The next example is also applied in Section 5.

Example 4.7. Let

$$\Psi_4^\gamma(u) := \tilde{G}(\tilde{G}^{-1}(u) + \gamma),$$

The FCDF (Ψ_4^γ) is similar to the WANG-transform but replacing the normal distribution function by the function

$$\tilde{G}(x) = 1 - \Gamma_{\alpha,\beta}\left(-\frac{\sqrt{\alpha}}{\beta}x\right), \quad x < 0,$$

where $\Gamma_{\alpha,\beta}$ is the gamma distribution with shape α and rate β. (Ψ_4^γ) generalizes the expected shortfall: for $\alpha = 1$ and $\beta = 1$, (Ψ_3^γ) and (Ψ_4^γ) are identical. Setting $\beta := \sqrt{\alpha}$, (Ψ_4^γ) converges to the WANG-transform for large α. We will see in Example 5.3, that the coherent risk measure induced by (Ψ_4^γ), reduces to the *standard deviation premium principle* when applied to gamma distributed random variables.

Cherny and Madan (2009) introduced four FCDF: the MAXVAR and MINVAR distortions, which are reparametrizations of the power distortion and its dual, the proportional hazards distortion, see Wang (1995, 1996), and the MINMAXVAR and MAXMINVAR, which are compositions of the former two.

None of those FCDF satisfies condition [T], but as we shall see, sometimes it is possible find a reparametrization, such that the reparameterized FCDF does satisfy condition [T] and hence can be represented by a distribution function. In the following definition we state more precisely what we mean by a reparametrization.

Definition 4.8. We say that the FCDF $\left(\tilde{\Psi}^\gamma\right)_{\gamma\geq 0}$ is a *reparametrization* of the FCDF $(\Psi^\gamma)_{\gamma\geq 0}$ if there exist bijective function

$$t : [0,\infty) \to [0,\infty)$$

such that $t(0) = 0$ and

$$\Psi^{t(\gamma)}(u) = \tilde{\Psi}^\gamma(u), \;\; u \in [0,1], \;\; \gamma \geq 0.$$

Example 4.9. The MAXVAR FCDF is defined by $\Psi^\gamma_{\mathrm{MAXVAR}}(u) = u^{\frac{1}{1+\gamma}}$ and there is a slight modification which indeed satisfies condition [T], in particular let

$$\tilde{\Psi}^\gamma_{\mathrm{MAXVAR}}(u) := u^{\exp(-\gamma)},$$

which is a reparametrization of $\Psi^\gamma_{\mathrm{MAXVAR}}$. By Theorem 4.1, the FCDF $\left(\tilde{\Psi}^\gamma_{\mathrm{MAXVAR}}\right)$ is induced by the distribution function

$$F_{\mathrm{MAXVAR}}(x) \;\; = \;\; e^{-\exp(-x)}, \;\; x \in \mathbb{R},$$

which is the Gumbel distribution with location zero and scale one.

Example 4.10. The MINVAR FCDF is defined by $\Psi^\gamma_{\mathrm{MINVAR}}(u) = 1-(1-u)^{\gamma+1}$ and can be represented after a reparametrization by $1 - G(-x)$, where G is the Gumbel distribution function with location zero and scale one. Let X be a Gumbel distributed random variable with location μ and scale $\sigma > 0$. X has distribution function

$$F_X(x) = e^{-\exp\left(-\frac{x-\mu}{\sigma}\right)}, \;\; x \in \mathbb{R}.$$

It holds

$$\rho_{\tilde{\Psi}^\gamma_{\mathrm{MINVAR}}}(X) = E[X] + \sigma\gamma$$

i.e. the coherent risk measures induced by the MINVAR FCDF and applied to a Gumbel distributed random variable X can be expressed by a linear mapping of the expectation of X.

We have seen in Example 4.9 and 4.10 that the MAXVAR and MINVAR FCDF defined by Cherny and Madan (2009) do not satisfy the condition [T] but there exist a reparametrization satisfying condition [T]. The following proposition is useful to check whether a FCDF can be reparameterized into a FCDF satisfying condition [T].

Proposition 4.11. *Let* (Ψ^γ) *be a FCDF. If there exist a reparametrization* $(\tilde{\Psi}^\gamma)$ *which satisfies condition [T], then it holds*

$$\Psi^{\gamma_1}\left(\Psi^{\gamma_2}(u)\right) = \Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u)\right), \;\; \gamma_1, \gamma_2 \geq 0, \;\; u \in [0,1], \tag{6}$$

i.e. the original FCDF is permutable.

Proof. Let $\gamma_1, \gamma_2 \geq 0$ and $u_0 \in [0,1]$. Then it follows

$$\Psi^{\gamma_1}\left(\Psi^{\gamma_2}(u_0)\right) = \tilde{\Psi}^{t(\gamma_1)}\left(\tilde{\Psi}^{t(\gamma_2)}(u_0)\right) = \tilde{\Psi}^{t(\gamma_1)+t(\gamma_2)}(u_0) = \Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u_0)\right),$$

for a suitable function t. □

Example 4.12. Simple numerical examples and Proposition 4.11 show that the following FCDF

$$\begin{aligned} \Psi^\gamma_{\text{MINMAXVAR}}(u) &= 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}, \\ \Psi^\gamma_{\text{MAXMINVAR}}(u) &= \left(1 - (1-u)^{\gamma+1}\right)^{\frac{1}{\gamma+1}}, \end{aligned}$$

cannot be reparameterized into a FCDF satisfying condition [T], i.e. cannot be represented by a distribution function.

5 Application: Coherent Risk Measures and Moment Based Premium Principles

A coherent risk measure ρ is a map from set of bounded random variables to the real numbers describing the riskiness of future random cash flows. In insurance science we are usually dealing with nonnegative random variables describing for example the possible financial loss due to a natural disaster. In an insurance context, we call a nonnegative random variable X *insurance risk* or just *risk* and the value $\rho(X)$ a *premium.*

It is possible to apply our representation result Theorem 4.1 to compare different insurance risks with each other. Let us assume an insurance company is insuring a risk, which can be described by a nonnegative random variable X. The amount of money charged by the insurer to the insured for the coverage of the loss due to the risk X, is called the *risk-adjusted premium*, excluding acquisition or internal expenses.

There are several method for assigning a risk-adjusted premium to the risk X. The premium could be defined via a coherent risk measure by $\rho(X)$. But many premium principles used in practice are equal to the expected value of the risk plus some security loading, so called *moment based premium principles*:

$$\begin{aligned} \text{the } \textit{Expected Value Premium} \text{ is defind by} \quad & E[X] + \gamma E[X], \\ \text{the } \textit{Standard Deviation Premium} \text{ is defind by} \quad & E[X] + \gamma\sqrt{\text{Var}(X)}, \\ \text{and the } \textit{Variance Premium} \text{ is defind by} \quad & E[X] + \gamma \text{Var}(X), \end{aligned}$$

where $\gamma \geq 0$, see Straub (1988), Daykin et al. (1994) and Rolski et al. (2009). The moment based premium principles are *not* coherent, the standard deviation premium principle for example is not monotone, i.e. two different risks cannot really be compared with each other[1]. But for a particular random variable X, it is possible to construct a FCDF (Ψ_X^γ), such that for a fixed $\xi > 0$ it holds

$$\rho_{\Psi_X^\gamma}(X) = E[X] + \gamma\xi, \quad \gamma \geq 0.$$

The value $\rho_{\Psi_X^\gamma}(X)$ is equal to a particular moment based premium of X for all $\gamma \geq 0$ if $\xi \in \{E[X], \sqrt{\mathrm{Var}(X)}, \mathrm{Var}(X)\}$. What are the benefits? An insurance which mainly insures a risk X and uses a moment based premium principle to assign a premium to X, might wish to compare risk X to another risk Z, which can be archived by comparing the values $\rho_{\Psi_X^\gamma}(X)$ and $\rho_{\Psi_X^\gamma}(Z)$ with each other.

On the one hand, the moment based premium principles are not coherent, they are arguably not very well suited to compare different risks with each other. They may even be infinite, e.g. if the second moments of Z do not exist.

On the other hand, moment based premium principles are easy to understand and explain to policyholders. That is why the insurance may use a moment based premium principle in the first place, to compute the premium of the risk X.

Note that already Wang (2000) observed that the WANG-transform leads to the standard deviation premium principle, if X is normal distributed. Our representation result for FCDF makes a straightforward computation of Ψ_X^γ possible, in particular for nonnegative and skewed random variables X.

5.1 Construction of a Coherent Risk Measure Reproducing a Moment-Based Premium Principle

In this section we construct a coherent risk measure, based on a concave distortion function and depending on a risk X, such that the premium principle of this risk measure reduces to the expected value, the standard deviation or the variance premium principle for risk X. Let an integrable, nonnegative random variable X on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. We make the following assumptions on the risk X:

Assumption 1. *The density f_X of X is continuous with support on $(0, \infty)$.*

Assumption 2. *The density f_X is log-concave.*

Assumption 3. *For the density it holds: $\lim_{x\to\infty} \frac{f_X(x-\gamma)}{f_X(x)} < \infty$ for all $\gamma > 0$.*

Those assumptions are made to keep the notation simple and could be relaxed. For example the densities of the normal distribution and the gamma, the beta

[1] For example let X take the values 10 or 90, each with the same probability. Clearly, X is less risky than the constant $Z = 100$. Let $\gamma = 1$. The standard deviation premium of X is about 106 but the premium of Z is smaller, it is equal to 100.

and the Weibull distribution, respectively with shape parameter $\alpha \geq 1$, are log-concave, see Bagnoli and Bergstrom (2005). Assumption 3 is used to show that a coherent risk measure induced by the distribution function of X is well defined on the whole space of integrable random variables L^1. In particular the gamma and the Weibull distributions satisfy assumptions $1-3$, both distributions are frequently used in insurance science to model insurance risks.

Proposition 5.1. *Let X satisfy Assumptions $1-3$. Let $\xi > 0$. Let*

$$G(x) := 1 - F_X(-x\xi), \;\; x \in \mathbb{R}.$$

The set of functions

$$\Psi_X^\gamma(u) = G(G^{-1}(u) + \gamma), \;\; \gamma \geq 0, \;\; u \in (0,1), \tag{7}$$

define a FCDF and it holds

$$\rho_{\Psi_X^\gamma}(X) = E[X] + \gamma\xi, \;\; \gamma \geq 0, \tag{8}$$

where $\rho_{\Psi_X^\gamma}$ is a coherent risk measure with domain L^1 induced by the concave distortion Ψ_X^γ, see Equation (2).

Remark 5.2. If $\xi \in \left\{E[X], \sqrt{\mathrm{Var}(X)}, \mathrm{Var}(X)\right\}$, the value $\rho_{\Psi_X^\gamma}(X)$ is then equal to the expected value premium, the standard deviation premium principle or the variance premium of X.

Proof. For $\gamma \geq 0$, we define Ψ_X^γ pointwise: $\Psi_X^\gamma(0) := 0$, $\Psi_X^\gamma(1) := 1$. Let $u \in (0,1)$ and let $x > 0$ such that

$$u = H(x) := 1 - F_X(x).$$

H is the decumulative distribution function of X. By Assumption 1, F_X is a bijective function from $(0,\infty)$ to $(0,1)$. It holds $x = H^{-1}(u)$ and we define

$$\Psi_X^\gamma(u) := H(H^{-1}(u) - \gamma\xi).$$

It follows

$$\Psi_X^\gamma(H(x)) = H(x - \gamma\xi), \;\; x > 0, \;\; \gamma \geq 0. \tag{9}$$

It is straightforward to see that $\gamma \mapsto \Psi_X^\gamma(u)$ is continuous and increasing and that $u \mapsto \Psi_X^\gamma(u)$ is increasing and concave, because the density corresponding to F_X is log-concave. Hence the family $(\Psi_X^\gamma)_{\gamma \geq 0}$ is a FCDF. It additionally satisfies conditions [E], [W] and [T], hence by Theorem 4.1, there exist a unique distribution function $\hat{G}$ such that $\hat{G}(0) = \frac{1}{2}$ and

$$\Psi_X^\gamma(u) = \hat{G}(\hat{G}^{-1}(u) + \gamma).$$

By Equation (5), $\hat{G}$ can be identified by

$$\hat{G}(x) = 1 - F_X\left(F_X^{-1}\left(\frac{1}{2}\right) - x\xi\right), \ \ x \in \mathbb{R}, \ \ \xi>0.$$

We shift $\hat{G}$ and define

$$G(x) = 1 - F_X(-x\xi)$$

and we have

$$\Psi_X^\gamma(u) = G(G^{-1}(u) + \gamma). \tag{10}$$

Let $g(x) := \xi f_X(-x\xi)$. It follows for $\gamma > 0$ by Assumption 3:

$$\begin{aligned}
\lim_{u \searrow 0} \frac{\partial}{\partial u} \Psi_X^\gamma(u) &= \lim_{u \searrow 0} \frac{g\left(G^{-1}(u) + \gamma\right)}{g\left(G^{-1}(u)\right)} \\
&= \lim_{x \to \infty} \frac{f_X(x - \xi\gamma)}{f_X(x)} < \infty.
\end{aligned}$$

Hence because Ψ_X^γ is concave for all $\gamma \geq 0$, its partial derivative is bounded on the unit interval and the coherent risk measures induced by the family (Ψ_X^γ) are well defined on L^1, see Remark 2.2. It follows by Equation (9) for all $\gamma \geq 0$

$$\begin{aligned}
E[X] + \gamma\xi &= \int_0^\infty 1 - F_X(x - \gamma\xi) dx \\
&= \int_0^\infty \Psi_X^\gamma(1 - F_X(x)) dx \\
&= \rho_{\Psi_X^\gamma}(X).
\end{aligned}$$

□

Example 5.3. Let $X \sim \Gamma(\alpha, \beta)$ be a gamma distributed random variable with mean $\frac{\alpha}{\beta}$ and variance $\frac{\alpha}{\beta^2}$ modelling a risk or an aggregated risk insured by the insurance company. The gamma distribution satisfies Assumption $1-3$, if $\alpha \geq 1$. We apply the standard deviation premium principle and choose

$$\xi = \sqrt{\mathrm{Var}(X)} = \frac{\sqrt{\alpha}}{\beta}.$$

Additionally, assume that the insurance faces another risk Z and wishes to compare both risks using a coherent risk measure, which reproduces the standard deviation premium for X and is induced by the FCDF (Ψ_X^γ), defined via Equation (7). Table 1 compares the standard deviation premium of X, to the premium of various other risks computed using $\rho_{\Psi_X^\gamma}$. The premium of a non-negative risk $Z \in L^1$ under $\rho_{\Psi_X^\gamma}$ is equal to

$$\rho_{\Psi_X^\gamma}(Z) = \int_0^\infty \Psi_X^\gamma(1 - F_Z(s)) ds. \tag{11}$$

The integral appearing in Equation (11) can be computed using standard numeric methods.

We compare risk X to an exponential, a Gaussian, a Bernoulli and a Pareto risk.

If $Z \sim \text{Pareto}(x_m, a)$ is Pareto distributed with scale $x_m > 0$ and shape $a > 0$ and if $a \in (1, 2]$, then Z has finite first and infinite second moments. In particular, the standard deviation premium principle cannot be applied to Z. The expected value of Z is $\frac{ax_m}{1-a}$ for $a > 1$. We further compare risk X to a risk W defined by the loss occurring in a layer with deductible $D \geq 0$ and cover $C > D$ of a Pareto distributed loss Z, i.e.

$$W := (Z - D)^+ - (Z - C - D)^+.$$

Let the distribution of W be denoted by

$$F_W^{\alpha, x_m, D, C}(x) := \begin{cases} 1 - \left(\frac{x_m}{x+D}\right)^\alpha & , (x_m - D, 0)^+ \leq x < C \\ 1 & , x \geq C. \end{cases}$$

It turns out that for $\gamma = 1$, the Standard Deviation Premia of the exponential and the Gaussian risk are very similar to the corresponding premia computed using $\rho_{\Psi_X^1}$. The differences between both premia for Bernoulli or Pareto risks are very large.

	X	Z_{exp}	Z_{Gauss}	Z_{B}	Z_∞	Z_{250}	Z_{10}
Expected Value	1	1	1	1	1	1	1
SD premium	1.47	2	1.20	10.95	∞	8.1	2.92
Premium under $\rho_{\Psi_X^1}$	1.47	1.99	1.19	4.25	4.31	3.46	2.64

Table 1: Compare the standard deviation (SD) premium principle to the premium principle using the coherent risk measure $\rho_{\Psi_X^1}$ applied to various risks: $X \sim \Gamma\left(\frac{9}{2}, \frac{9}{2}\right)$, $Z_{\text{exp}} \sim \exp(1)$, $Z_{\text{Gauss}} \sim N(1, \frac{2}{10})$, Z_B is Bernoulli distributed taking the value 100 with probability $\frac{1}{100}$. $Z_\infty \sim \text{Pareto}(\frac{1}{10}, \frac{10}{9})$, $Z_{250} \sim F_W^{\frac{10}{9}, 0.2, 0.2, 250}$ and $Z_{10} \sim F_W^{\frac{10}{9}, 0.36, 0.36, 10}$. The concave distortion function Ψ_X^1 is drawn in Figure 1 as Ψ_4.

5.2 Interpretation of the Coherent Risk Measure $\rho_{\Psi_X^\gamma}$

Recently Cherny and Madan (2009) provided an axiomatic approach to study performance measures in a unified way. They defined an *acceptability index* $\alpha : L^\infty \to [0, \infty]$ as a monotone, quasi-concave, scale-invariant and semi-continuous map assigning to a terminal cash flow a positive value. The higher that value, the more attractive is the position. A famous example is the gain-loss ratio, see Bernardo and Ledoit (2000).

As above let X describe some insurance risk and let π_X be the premium of X obtained by a moment based premium principle. Let the FCDF (Ψ_X^γ) be defined such that $\pi_X = \rho_{\Psi_X^\gamma}(X)$. The following proposition offers an interpretation of the premium principle based on the coherent risk measure $\rho_{\Psi_X^\gamma}$. There is an acceptability index α such that the performance of the future random cash flow

$$\rho_{\Psi_X^\gamma}(Z) - Z$$

for any risk $Z \in L^1$ is at least as high as the performance of the cash flow $\pi_X - X$. Using only the acceptability index α as a criterion, the insurance is indifferent insuring risk X and obtaining premium π_X or insuring another risk Z in return for premium $\rho_{\Psi_X^\gamma}(Z)$.

Proposition 5.4. *Let X satisfy Assumptions $1-3$. For some $\xi > 0$, let the FCDF $(\Psi_X^\gamma)_{\gamma \geq 0}$ be defined as in Equation (7). Let $\gamma_0 \geq 0$ and*

$$\pi_X := E[X] + \gamma_0 \xi.$$

There exist an acceptability index $\alpha : L^1 \to [0, \infty]$ such that

$$\alpha(\pi_X - X) = \gamma_0 \leq \alpha\left(\rho_{\Psi_X^{\gamma_0}}(Z) - Z\right), \tag{12}$$

for all $Z \in L^1$ with $Z \geq 0$.

By convention, the performance of the null-position is infinite. Therefore the right-hand side of Equation (12) can be equal to infinity, for example if $Z = 0$.

Proof. The family of coherent risk measures $\left(\rho_{\Psi_X^\gamma}\right)_{\gamma \geq 0}$ has domain L^1 and defines an acceptability index α by

$$\begin{aligned} \alpha : L^1 &\to [0, \infty] \\ Y &\mapsto \sup\left\{\gamma \geq 0 : \rho_{\Psi_X^\gamma}(-Y) \leq 0\right\}, \end{aligned}$$

see Cherny and Madan (2009, eq. (4)) and Remark 2.3. Let $Z \in L^1$ such that $Z \geq 0$. It holds using the translation property for coherent risk measures

$$\begin{aligned} \alpha\left(\rho_{\Psi_X^{\gamma_0}}(Z) - Z\right) &= \sup\left\{\gamma \geq 0 : \rho_{\Psi_X^\gamma}\left(-\left(\rho_{\Psi_X^{\gamma_0}}(Z) - Z\right)\right) \leq 0\right\} \\ &= \sup\left\{\gamma \geq 0 : \rho_{\Psi_X^\gamma}(Z) \leq \rho_{\Psi_X^{\gamma_0}}(Z)\right\} \\ &\geq \gamma_0 \end{aligned}$$

and similarly

$$\alpha(\pi_X - X) = \sup\left\{\gamma \geq 0 : \rho_{\Psi_X^\gamma}(X) \leq E[X] + \gamma_0 \xi\right\} = \gamma_0.$$

□

6 Conclusion

In this article we pointed out the relation between a family of concave distortion function (FCDF) and coherent risk measures. A concave distortion function is a concave function mapping the unity interval onto itself. A coherent risk measures can be defined by distorting the original distribution function of a random variable: losses are given more weight and gains are given less weight. We have shown that a FCDF satisfying a certain translation equation, can be represented by a distribution function. Our representation theorem is novel, it generalizes a comparable result obtained by Tsukahara (2009).

In contrast to Tsukahara (2009), our representation results also covers FCDF which are not strictly increasing in the distortion level like the FCDF related to the expected shortfall and FCDF which jump like the "ess sup-*expectation convex combination*" distortion function defined and applied to finance by Bannör and Scherer (2014).

On the other hand, Tsukahara's result does not require the family of distortion functions to be concave. But concavity is a natural requirement when dealing with coherent risk measures. A risk measure should encourage diversification, i.e. the risk of a portfolio must not exceed the sum of the risk of its components. A risk measures induced by a distortion function which is not concave, is in general not sub-additive and does not encourage diversification.

An application of the representation result can be found in actuarial science: assume there is an insurance company selling mainly contracts to insure a risk X. The risk X may describe a loss due to some natural disaster like fire. The insurance company computes the premium of the insurance contract using a moment based premium principle, e.g. the premium is calculated as the expected value of X plus a multiple of the standard deviation of X. Such a premium principle is easy to understand and to explain to policyholders but it is not monotone, i.e. different insurance risks cannot be compared with each other and cannot be priced in a consistent way.

Our representation theorem makes it possible to construct a coherent risk measure ρ_X, induced by a concave distortion function and depending on the distribution function of X, such that the premium principle of that risk measure reduces to a moment based premium principle when applied to risk X. The price of another insurance risk Z may then be compared to the standard deviation premium of X, even if the variance of Z does not exist, by applying ρ_X both to X and to Z.

The premium principle based on ρ_X is consistent with a moment based premium principle like the standard deviation premium principle. The residual cash flow of the insurance company insuring risk X in return for the (standard deviation premium) is the difference of the premium and the insurance risk X. We show that there exists an acceptability index (performance measure) such that the performance of the residual cash flow insuring risk X is equal to the performance of the residual cash flow insuring any other risk Z, if the premium of Z is computed based on ρ_X.

Using only this acceptability index as a criterion, the insurance is indifferent insuring risk X and obtaining a standard deviation premium or insuring another risk Z in return for the premium $\rho_X(Z)$.

7 Appendix

The following lemma shows that a FCDF can only be represented by a distribution function G with a certain structure, e.g. G is continuous on the whole real line and strictly increasing on its support until it hits its upper limit 1.

Lemma 7.1. *Let $u_0 \in (0,1)$. Let $G : \mathbb{R} \to [0,1]$ be a distribution function such that $G(0) = u_0$. Define $G(-\infty) = 0$. Let G^{-1} be the generalized inverse of G, for instance*

$$G^{-1}(u) := \inf\{x \in \mathbb{R} \,:\, G(x) \geq u\}\,.$$

Define $x_0 := \inf\{x \in \mathbb{R},\ G(x) > 0\}$ and $x_1 := G^{-1}(1)$. It then holds $x_0 < x_1$. Let (Ψ^γ) be a FCDF. If

$$\Psi^\gamma(u) = G(G^{-1}(u) + \gamma),\ \ u \in (0,1),\ \ \gamma \geq 0,$$

then it holds $G(x_0) = 0$ and G is continuous on $\mathbb{R}$ and strictly increasing on (x_0, x_1). We further have

$$G^{-1}(G(x)) = x,\ \ x \in (x_0, x_1) \tag{13}$$

and

$$G(G^{-1}(u)) = u,\ \ u \in (0,1). \tag{14}$$

Proof. We trivially have $x_0 \leq 0 < x_1$. Assume $0 < p_0 := G(x_0)$. Then $p_0 \leq u_0 < 1$ and $G^{-1}(p) = x_0$ for $p \in (0, p_0]$. Hence the map $u \mapsto G(G^{-1}(u))$ is constant and equal to p_0 on $(0, p_0)$, which is a contraction as the map $u \mapsto \Psi^0(u)$ is concave and increasing and $\Psi^0(1) = 1$. Thus it holds $G(x_0) = 0$.

As G is a distribution function, G is right-continuous and increasing, i.e. for all $x \in \mathbb{R}$ it holds

$$G(x+) := \lim_{\varepsilon \downarrow 0} G(x + \varepsilon) = G(x).$$

Assume there is a $\bar{x} \in (x_0, x_1]$ such that

$$\bar{u} := G(\bar{x}-) := \lim_{\varepsilon \uparrow 0} G(\bar{x} + \varepsilon) < G(\bar{x})$$

i.e. G jumps at $\bar{x}$. Then $G(G^{-1}(\bar{u}-)) < G(\bar{x}) \leq G(G^{-1}(\bar{u}+))$, which is a contradiction because the map $u \mapsto \Psi^0(u)$ is continuous on $(0,1]$. We conclude that G is continuous on $\mathbb{R}$.

Now we show, that G is strictly increasing on (x_0, x_1). Assume there are $x_0 < \tilde{x}_1, \tilde{x}_2 < x_1$ such that $\tilde{x}_1 < \tilde{x}_2$ and $G(\tilde{x}_1) = G(\tilde{x}_2) =: \tilde{u}$. Then it follows $0 < \tilde{u} < 1$ and there exists $\gamma > 0$ such that

$$G(G^{-1}(\tilde{u}-) + \gamma) \leq G(\tilde{x}_1 + \gamma) < G(\tilde{x}_2 + \gamma) \leq G(G^{-1}(\tilde{u}+) + \gamma),$$

which is again a contraction. The second assertion, expressed by Equations (13) and (14), follows immediately, because $\tilde{G} : (x_0, x_1) \to (0,1)$, $x \mapsto G(x)$, is bijective. ☐

Prove of Theorem 4.1. We show the direction i)⇒ ii). Let $u_0 \in (0,1)$ and define $G : \mathbb{R} \to [0,1]$ by Equation (5).

First step: Show that $p \mapsto \overline{\Psi}^{\gamma}(p)$ is continuous.

By Definition 3.1, for a fixed $\gamma \geq 0$, the function $u \mapsto \Psi^{\gamma}(u)$ is monotonically increasing and concave and it holds $\Psi^{\gamma}(0) = 0$ and $\Psi^{\gamma}(1) = 1$. This implies a strong structure on Ψ^{γ}: There exists a constant $\tilde{u}_{\gamma} \in [0,1]$, namely

$$\tilde{u}_{\gamma} = \inf \left\{ u \, : \, \Psi^{\gamma}(u) = 1 \right\}, \tag{15}$$

such that $u \mapsto \Psi^{\gamma}(u)$ is strictly increasing and continuous on $(0, \tilde{u}_{\gamma}]$ and constant on $(\tilde{u}_{\gamma}, 1]$. At zero, $u \mapsto \Psi^{\gamma}(u)$ might jump. Let $\tilde{p}_{\gamma} := \lim_{\varepsilon \downarrow 0} \Psi^{\gamma}(\varepsilon)$ be the jump-size at $u = 0$. For a particular distortion function, $\tilde{u}_{\gamma}$ and $\tilde{p}_{\gamma}$ are visualized in Figure 1. By definition of $p \mapsto \overline{\Psi}^{\gamma}(p)$, it holds for $0 \leq p \leq \tilde{p}_{\gamma}$

$$\overline{\Psi}^{\gamma}(p) = \inf \left\{ u \in [0,1] \, : \, \Psi^{\gamma}(u) \geq p \right\} = \inf \left\{ u \in (0,1] \right\} = 0. \tag{16}$$

Continuity of $p \mapsto \overline{\Psi}^{\gamma}(p)$ follows immediately: define

$$\Theta^{\gamma}(u) := \begin{cases} \tilde{p}_{\gamma} & , u = 0 \\ \Psi^{\gamma}(u) & , u > 0. \end{cases}$$

Then $u \mapsto \Theta^{\gamma}(u)$ is continuous and bijective as a function from $[0, \tilde{u}_{\gamma}]$ to $[\tilde{p}_{\gamma}, 1]$ and hence its inverse $\overline{\Theta}^{\gamma}$ is also continuous. We further have $\overline{\Psi}^{\gamma}(p) = \overline{\Theta}^{\gamma}(p)$ for $p \in [\tilde{p}_{\gamma}, 1]$, which shows continuity of $p \mapsto \overline{\Psi}^{\gamma}(p)$.

Second step: show that $\gamma \mapsto \overline{\Psi}^{\gamma}(u_0)$ is decreasing and continuous, hence G is a distribution function.

While $\gamma \mapsto \Psi^{\gamma}(u_0)$ is increasing and continuous in the variable γ by definition, it is easy to see that its generalized inverse is decreasing in the variable γ. The function $\gamma \mapsto \overline{\Psi}^{\gamma}(u_0)$ is continuous, which can be seen by the following auxiliary result:

If $\gamma_2 \geq \gamma_1 \geq 0$ and $\Psi^{\gamma_2 - \gamma_1}(u_0) < 1$ and $u_0 > \tilde{p}_{\gamma_1}$, it follows

$$\Psi^{\gamma_2 - \gamma_1}(u_0) = \Psi^{\gamma_2 - \gamma_1} \left(\Psi^{\gamma_1} \left(\overline{\Psi}^{\gamma_1} (u_0) \right) \right) = \Psi^{\gamma_2} \left(\overline{\Psi}^{\gamma_1} (u_0) \right).$$

Applying $\overline{\Psi}^{\gamma_2}$ on both sides, yields

$$\overline{\Psi}^{\gamma_1}(u_0) = \overline{\Psi}^{\gamma_2} \left(\Psi^{\gamma_2 - \gamma_1}(u_0) \right). \tag{17}$$

Let $\gamma_0 := \inf \{ \gamma \geq 0 : \tilde{p}_{\gamma} \geq u_0 \}$, where $\inf \emptyset = \infty$. γ_0 is the smallest number, such that the jump-size of Ψ^{γ_0} at zero is greater or equal to u_0. The map $\gamma \mapsto \overline{\Psi}^{\gamma}(u_0)$ is identical to zero on $[\gamma_0, \infty)$, compare with Equation (16). It remains to show continuity from below at $\gamma \in (0, \gamma_0]$ and continuity from above at $\gamma \in (0, \gamma_0)$. Let $0 < \gamma \leq \gamma_0$ and $(\gamma_n)_{n\in\mathbb{N}}$ be a positive sequence converging from below to γ. Without loss of generality, we assume $\gamma_n < \gamma$ for all n. For n large enough, it holds $\Psi^{\gamma-\gamma_n}(u_0) < 1$ because Ψ^γ is continuous at γ and $\Psi^0(u_0) = u_0 < 1$. We have $u_0 > \tilde{p}_{\gamma_n}$ because $\gamma_n < \gamma_0$ and by Equation (17), it holds

$$\overline{\Psi}^{\gamma_n}(u_0) = \overline{\Psi}^{\gamma}\left(\Psi^{\gamma-\gamma_n}(u_0)\right) \to \overline{\Psi}^{\gamma}(u_0), \;\; n \to \infty,$$

where we used that $p \mapsto \overline{\Psi}^{\gamma}(p)$ is continuous on $[0,1]$. If $\gamma < \gamma_0$ and (γ_n) is a sequence converging from above to γ, let $\varepsilon > 0$ such that $\Psi^{\varepsilon\gamma}(u_0) < 1$ and choose n large enough so that $(1+\varepsilon)\gamma - \gamma_n \geq 0$ and $\gamma_n < \gamma_0$. It follows $\Psi^{(1+\varepsilon)\gamma-\gamma_n}(u_0) < 1$ and using Equation (17) twice and continuity of $p \mapsto \overline{\Psi}^{\gamma}(p)$, shows continuity from above.

Thus G is monotonically increasing and continuous. Continuity at zero can be shown using condition [E]: it holds $G(0) = \Psi^0(u_0) = u_0$. By condition [W] it follows $\lim\limits_{x\to\infty} G(x) = 1$ and $\lim\limits_{x\to-\infty} G(x) = 0$. G is thereby a distribution function.

Third step: show that Equation (4) holds.

We distinguish three cases and use that $(\Psi^\gamma)_{\gamma\geq 0}$ satisfies condition [T]. Let $\gamma \geq 0$ and $u \in (0,1)$. As G is continuous, it is a surjective function from $\mathbb{R}$ to $(0,1)$ and there exists $x \in \mathbb{R}$ such that $G(x) = u$ and $G^{-1}(u) = x$. If $x \geq 0$, it follows

$$\begin{aligned} G(x+\gamma) &= \Psi^{x+\gamma}(u_0) \\ &= \Psi^{\gamma}\left(\Psi^{x}(u_0)\right) \\ &= \Psi^{\gamma}(G(x)). \end{aligned}$$

If $x < 0$, it holds $\overline{\Psi}^{-x}(u_0) = G(x) > 0$ and therefore $u_0 > \tilde{p}_{-x}$. If $x < 0$ and $x + \gamma \geq 0$, it follows

$$\begin{aligned} G(x+\gamma) &= \Psi^{x+\gamma}(u_0) \\ &= \Psi^{x+\gamma}\left(\Psi^{-x}\left(\overline{\Psi}^{-x}(u_0)\right)\right) \\ &= \Psi^{\gamma}(G(x)). \end{aligned}$$

If $x < 0$ and $x + \gamma < 0$ we have

$$1 > u_0 = \Psi^{-x}\left(\overline{\Psi}^{-x}(u_0)\right) = \Psi^{-\gamma-x}\left(\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right)\right)$$

and thereby $\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right) < \tilde{u}_{-\gamma-x}$, compare with Equation (15). We further have $\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right) > 0$ as $\overline{\Psi}^{-x}(u_0) = G(x) = u > 0$. Because $\Psi^{-\gamma-x} : (0, \tilde{u}_\gamma] \to (\tilde{p}_\gamma, 1]$ is bijective, it follows

$$\begin{aligned} G(x+\gamma) &= \overline{\Psi}^{-x-\gamma}(u_0) \\ &= \overline{\Psi}^{-x-\gamma}\left(\Psi^{-\gamma-x}\left(\Psi^{\gamma}\left(\overline{\Psi}^{-x}(u_0)\right)\right)\right) \\ &= \Psi^{\gamma}(G(x)). \end{aligned}$$

Fourth step: Show the uniqueness of G.

Let assume there is another distribution function F such that $F(0) = u_0$ and

$$F(F^{-1}(u) + \gamma) = \Psi^{\gamma}(u), \ \ u \in (0,1), \ \ \gamma \geq 0.$$

For $x \geq 0$ it follows by Lemma 7.1,

$$F(x) = F(F^{-1}(u_0) + x) = \Psi^{x}(u_0) = G(x).$$

Let $x_0 := \inf\{x, F(x) > 0\}$. For $x_0 < x < 0$, it follows $0 < F(x) < 1$ and it holds

$$\Psi^{-x}(F(x)) = F(F^{-1}(F(x)) - x) = F(0) = u_0$$

and hence

$$F(x) = \overline{\Psi}^{-x}(u_0) = G(x).$$

If $-\infty < x_0$, we further have

$$\tilde{p}_{-x_0} = \lim_{\varepsilon \downarrow 0} F(F^{-1}(\varepsilon) - x_0) = F(0) = u_0$$

and therefore $G(x_0) = \overline{\Psi}^{-x_0}(u_0) = 0 = F(x_0)$. Hence it holds $G(x) = F(x)$ for all $x \in \mathbb{R}$.

Now let us show the other direction ii)$\Rightarrow$ i). We use lemma 7.1. Let $u_0 \in (0,1)$. If there is a distribution function G such that $G(0) = u_0$ and Equation (4) holds, it follows for any $u \in (0,1]$

$$\lim_{\gamma \to \infty} \Psi^{\gamma}(u) = \lim_{\gamma \to \infty} G(G^{-1}(u) + \gamma) = 1,$$

i.e. (Ψ^{γ}) satisfies condition [W]. We further have

$$\Psi^{0}(u) = G(G^{-1}(u)) = u, \ \ u \in (0,1),$$

which shows that the FCDF satisfies condition [E]. Now let $\gamma_1, \gamma_2 \geq 0$ and $u \in (0,1)$. Assume $\Psi^{\gamma_1}(u) < 1$, then it holds

$$\begin{aligned}
\Psi^{\gamma_2}\left(\Psi^{\gamma_1}(u)\right) &= G(G^{-1}\left[G(G^{-1}(u) + \gamma_1)\right] + \gamma_2) \\
&= G(G^{-1}(u) + \gamma_1 + \gamma_2) \\
&= \Psi^{\gamma_1 + \gamma_2}(u).
\end{aligned}$$

The case $\Psi^{\gamma_1}(u) = 1$ is trivial. Thus (Ψ^{γ}) satisfies condition [T]. ☐

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