Cost-efficient payoffs under model ambiguity

Carole Bernard[*]
Gero Junike[†]
Thibaut Lux[‡]
Steven Vanduffel[§¶]

October 7, 2021

Abstract

A payoff that is the cheapest possible in reaching a given target distribution is called cost-efficient. In the presence of ambiguity the distribution of a payoff is no longer known. A payoff is called robust cost-efficient if its worst-case distribution stochastically dominates a target distribution and is the cheapest possible in doing so. We study the link between this notion of "robust cost-efficiency" and the maxmin expected utility setting of Gilboa and Schmeidler, as well as more generally with robust preferences in a possibly non-expected utility setting. We illustrate our study with examples involving uncertainty both on the drift and on the volatility of the risky asset.

KEYWORDS: Cost-efficient payoffs, model ambiguity, maxmin utility, robust preferences, drift and volatility uncertainty
JEL classification: C02, C63, D80
AMS classification: 91B30, 62E17

[*] Corresponding author: Carole Bernard, Department of Accounting, Law and Finance at Grenoble Ecole de Management (email: carole.bernard@grenoble-em.com) and Department of Economics and Political Sciences at Vrije Universiteit Brussel (VUB).

[†] Institute of Mathematics, Carl von Ossietzky University, 26111 Oldenburg, Germany (e-mail: gero.junike@uol.de).

[‡] Helvetia Insurance Group (e-mail: thibaut.lux@helvetia.ch). *The views expressed in this manuscript do not represent the opinions of Helvetia Insurance Group.

[§] Steven Vanduffel, Department of Economics and Political Sciences at Vrije Universiteit Brussel (VUB). (e-mail: steven.vanduffel@vub.ac.be).

[¶] We thank Ruodu Wang for constructive comments on an earlier version. This work was supported by the FWO Odysseus project FWOODYS11 of Carole Bernard at VUB.

1 Introduction

Uncertainty has become a prime issue in many academic domains, from economics to environmental science to psychology. This development stems, on the one hand, from the pervasive nature of uncertainty, as it permeates most aspects of everyday life, and on the other hand, it is driven by the fact that uncertainty has − sometimes profound − consequences regarding decisions, strategies and interactions among agents. In the economics literature, uncertainty takes its most prominent form in terms of model ambiguity. The probabilistic notion of model ambiguity refers to random phenomena or outcomes whose probabilities are themselves unknown. For instance, the random outcome of a coin toss is subject to model uncertainty when the probability of the coin showing either a head or a tail is not or is at most partially known. This notion of model ambiguity goes back to Knight (1921) and is therefore commonly referred to as Knightian uncertainty. In this paper, we study the impact of Knightian uncertainty on optimal choice among financial payoffs. In a world without ambiguity, **?**Dybvig (1988) characterizes optimal payoffs for agents having law-invariant increasing preferences. His analysis is based on the observation that any optimal payoff X is cost-efficient in the sense that there cannot exist another payoff with the same probability distribution that is strictly cheaper than X. He then derives, for a given target distribution of terminal wealth, the payoff that achieves this target distribution at the lowest possible cost.

We study the cost-minimization problem for an agent who evaluates payoffs against multiple subjective probability measures $\mathbb{P}$ in a class $\mathcal{P}$ of measures for the underlying market. The set $\mathcal{P}$ can be thought of as a collection of probabilities that are considered possible or plausible models for the market mechanism. The agent then seeks to construct a payoff at minimal cost, which is universally preferred to a benchmark distribution under all plausible measures in $\mathcal{P}$, i.e., which is such that for every measure its distribution dominates the target distribution w.r.t. some stochastic integral order. We provide an explicit solution to the optimization problem under certain conditions on the set $\mathcal{P}$. The solution is based on the identification of a least favourable reference measure $\mathbb{P}^* \in \mathcal{P}$ with respect to the stochastic integral order at hand, as it shows that this reduces the problem to the standard cost-efficiency problem in a market without uncertainty, governed by a single measure $\mathbb{P}^*$. Our result generalizes the results given in **?**Dybvig (1988), Cox and Leland (2000) and Bernard et al. (2014, 2015) to a setting of model ambiguity. We illustrate our results in numerical examples related to optimal investments in log-normal and Lévy markets with unknown parameters.

Optimal payoff choice under ambiguity includes in its pedigree Gilboa and Schmeidler (1989) "maxmin expected utility" with non-unique priors. These authors characterize preference relations that have a robust utility numerical representation $\min_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}(u(X))$. Gundel (2005) provides a dual characterization for the solution for robust utility maximization both in a complete and an incomplete market model. Klibanoff et al. (2005) distinguish between subjective beliefs, e.g. the definition of the set of possible or plausible subjective probability measures, and ambiguity attitude, i.e. a characterization of the agent's behavior towards ambiguity. Based on Klibanoff et al. (2005), Gollier (2011) analyses the effect of ambiguity aversion on the demand for the uncertain asset in a portfolio choice problem.

Schied (2005) solves the maximization problem of maxmin expected utility in a general complete market model with dynamic trading, provided there is a *least favourable measure* with respect to FSD-order. Specifically, he finds that the optimum for the maxmin utility setting of Gilboa and Schmeidler (1989) can be derived in the standard expected utility setting under the least favorable measure. While Schied (2005) assumes a time dynamic setup, we consider a static setting. Moreover, Schied works in complete markets while we have incomplete markets and are thus able to address uncertainty about volatility without introducing arbitrage. We show that the solution to a general robust portfolio maximization problem is equal to the solution of a classical portfolio maximization problem under a least favourable measure $\mathbb{P}^*$ with respect to some stochastic order. In our setting, we show that for general preferences, the result that Schied established in the context of expected utility theory still holds.

Furthermore, we show that there is a natural correspondence between optimal portfolios in the maxmin utility setting of Gilboa and Schmeidler with a concave increasing utility and robust cost-efficient payoffs: for any robust cost-efficient payoff X^*, there is a (generalized) utility function such that X^* solves the maxmin expected utility maximization problem. We further show that the solution to a robust maximization problem with respect to a general family of preferences is cost-efficient. This result implies that instead of solving a robust maximization problem with respect to a general family of preferences one could solve an expected utility maximization problem under the single measure $\mathbb{P}^*$ for a suitable concave utility function.

The paper is organized as follows. In Section 2, we present the market setting and our assumptions. In particular, in Section 2.1, we show that any derivative satisfying some mild technical assumption can be approximated by a static portfolio of plain vanilla call options. In Section 3, we introduce and solve the robust cost-efficiency problem, and we end this section with two examples in a log-normal market with uncertainty on the drift and the volatility along with another example in a Lévy market in which the physical measure is obtained by the Esscher transform. In Section 4, we develop the correspondence between robust cost-efficient payoffs and strategies that solve a robust optimal portfolio problem, including the maxmin utility setting of Gilboa and Schmeidler (1989) as a special case. In Section 5, we show that the solution to a general robust optimal portfolio problem can also be obtained as a solution to the maximization of the maxmin utility setting of Gilboa and Schmeidler (1989) for a well chosen concave utility function. Section 6 concludes.

2 Market setting

We assume a static setting, in which trading takes place only today and at maturity $T > 0$. There is a bank account earning the risk-free interest rate $r \in \mathbb{R}$. Let $S_T : \Omega \to \mathbb{R}_+$ represent the random value of a risky asset at maturity. Its value today is $S_0 < \infty$. Let $\mathcal{F}$ be the σ-algebra generated by S_T and denote by $\mathcal{X}$ the set of all measurable functions on $(\Omega, \mathcal{F})$, i.e., for any $X \in \mathcal{X}$, there is a measurable map $g : \mathbb{R}_+ \to \mathbb{R}$ such that $X = g(S_T)$. Any such X will be called a payoff.

Let $\mathcal{P}$ be a set of equivalent real-world probability measures on $(\Omega, \mathcal{F})$. The set $\mathcal{P}$ can be thought of as a collection of probability measures that the agent deems plausible for the distribution of S_T. We assume that there is an equivalent martingale measure $\mathbb{Q}$ (so that there is no arbitrage), and from now onwards all payoffs are already discounted. We also assume that S_T has a bounded density $f^{\mathbb{Q}}_{S_T}$ with support on $[0, \infty)$ under any equivalent martingale measure $\mathbb{Q}$.

Remark 2.1. To assume that for every measure $\mathbb{Q}$, S_T has a Lebesgue density is a weak assumption. For many stock price models - like the Heston or Bates model, see Heston (1993) and Bates (1996), the Variance-Gamma model, see Madan et al. (1998) or the BNS model, see Barndorff-Nielsen and Shephard (2001) - the characteristic function φ_Z of $Z = \log(S_T)$ is known and integrable. The density f_Z of Z exists and is continuous, if φ_Z is integrable. In this case, f_Z can be retrieved from φ_Z by the Fourier transform. An efficient numerical procedure to compute f_Z is described in Fang and Oosterlee (2009) and Junike and Pankrashkin (2021). The density f_{S_T} of S_T is then continuous and given by $f_{S_T}(x) = f_Z(\log(x))\frac{1}{x}$ for $x > 0$.

2.1 Payoffs with known market price

We denote by $\mathcal{X}_e \subset \mathcal{X}$ the set of all payoffs X having a unique market price $\pi(X)$. It holds that

$$\pi(X) = E_{\mathbb{Q}}[X].$$

In particular, $S_T \in \mathcal{X}_e$ and $\pi(S_T) = S_0$. In the remainder of this article we assume that for any $K \geqslant 0$, the call option written on S_T and having strike K is traded and thus is included in $\mathcal{X}_e$.

In this regard, Carr and Madan (1998) argue that the assumption of the existence of a continuum of strikes "is essentially the analog of the standard assumption of continuous trading. Just as the latter assumption is frequently made as a reasonable approximation to an environment where agents can trade frequently, [the assumption that there is a continuum of strikes] is a reasonable approximation when there are a large but finite number of option strikes (e.g. for S&P500 futures options)".

We show in Proposition 2.2 hereafter that the assumption that all call options are traded suffices to show that, subject to a technical condition of square integrability, any payoff $g(S_T)$ is included in $\mathcal{X}_e$. The proof is relegated to Appendix A

Proposition 2.2. *Let $g : [0, \infty) \to \mathbb{R}$ be square-integrable on any compact interval (for the Lebesgue measure) and $E_{\mathbb{Q}}[|g(S_T)|] < \infty$. Then $g(S_T) \in \mathcal{X}_e$.*

Example 2.3. If g is piecewise continuous with bounded jumps, then g is square integrable on any compact interval.

The proof of Proposition 2.2 also shows that a square integrable derivative $X = g(S_T)$ can be replicated by a portfolio of possibly infinitely many call options in the sense that a portfolio converging to $g(S_T)$ in mean (L^1) can always be constructed.

Similar versions of Proposition 2.2 exist in the literature. Breeden and Litzenberger (1978) show that when g is measurable, the payoff $g(S_T)$ is attainable (and thus has a unique price) if the function mapping every $K \geqslant 0$ to the price of the call with strike K is twice differentiable with respect to K. This can be considered as a strong condition in that it implies that linear interpolation between call prices is not allowed. Carr and Madan (1998) proved that $g(S_T)$ is attainable if g is twice differentiable. Even the payoff functions of a digital option or of a call option are not twice differentiable, although they could be approximated by a smooth function. Nachman (1988, Theorem 4) shows that "call options on portfolios of call options on individual primitive securities approximately span the entire completion of the primitive securities". See also Green and Jarrow (1987), who worked in a similar vein. In contrast to the above authors, who consider almost sure convergence, we consider convergence in mean. The idea of measuring the difference between a contingent claim and its replicating portfolio by a L^P-norm, moreover, is not new; for an early reference see Duffie and Richardson (1991), who minimized the mean-square hedging-error in incomplete markets.

2.2 Likelihood ratio

The likelihood ratio between a probability measure $\mathbb{P} \in \mathcal{P}$ and the risk-neutral probability measure $\mathbb{Q}$ will be used to construct cost-efficient payoffs and is defined next.

Definition 2.4. The Radon–Nikodym derivative $\ell^{\mathbb{P}} = \frac{d\mathbb{P}}{d\mathbb{Q}}$ for $\mathbb{P} \in \mathcal{P}$ is called *likelihood ratio*, i.e., $\ell^{\mathbb{P}}$ satisfies

$$\forall A \in \mathcal{F},\ \forall \mathbb{P} \in \mathcal{P}, \quad \mathbb{P}(A) = \int_A \ell^{\mathbb{P}} d\mathbb{Q}.$$

The likelihood ratio can be used to express the price of any $X \in \mathcal{X}_e$ under the real-world probability measure $\mathbb{P}$ as

$$\pi(X) = E_{\mathbb{Q}}[X] = E_{\mathbb{P}}\left[\frac{X}{\ell^{\mathbb{P}}}\right].$$

Remark 2.5. Instead of the likelihood ratio, one could work with the *state-price*, defined by $\xi^{\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}}$ for $\mathbb{P} \in \mathcal{P}$. The choice to work with the likelihood ratio or the state-price is a matter of taste and has no further mathematical implications. If the state-price is used to express the price of a payoff X, then $E_{\mathbb{Q}}[X] = E_{\mathbb{P}}[X\xi^{\mathbb{P}}]$.

Proposition 2.6. *It holds*

$$\ell^{\mathbb{P}} = \frac{f^{\mathbb{P}}_{S_T}(S_T)}{f^{\mathbb{Q}}_{S_T}(S_T)}, \ \ \mathbb{P} - a.s.$$

Proof. Let $A \in \mathcal{F}$. Then there is a Borel-measurable function $g : [0,\infty) \to \mathbb{R}$ such that $1_A = g(S_T)$. It follows that

$$\begin{aligned}\mathbb{P}(A) &= \int_\Omega 1_A d\mathbb{P} \\ &= \int_{[0,\infty)} g(s) \frac{f^{\mathbb{P}}_{S_T}(s)}{f^{\mathbb{Q}}_{S_T}(s)} f^{\mathbb{Q}}_{S_T}(s) ds \\ &= \int_A \frac{f^{\mathbb{P}}_{S_T}(S_T)}{f^{\mathbb{Q}}_{S_T}(S_T)} d\mathbb{Q}.\end{aligned}$$

The Radon-Nikodym derivative is $\mathbb{P}-$a.s. unique, which finishes the proof. ☐

Applying Proposition 2.6, we can express the likelihood ratio by $g(S_T)$ for some function g. We assume that the densities of S_T under $\mathbb{P}$ and $\mathbb{Q}$ are such that g satisfies the assumption of Proposition 2.2, and hence $\ell^{\mathbb{P}}$ can be replicated by a portfolio of call options. Throughout the paper, optimal payoffs are expressed as functions of the likelihood ratio, $h(\ell^{\mathbb{P}})$, for piecewise continuous functions h. We also assume that $h \circ g$ satisfies the assumption of Proposition 2.2 and hence that $h(\ell^{\mathbb{P}})$ can be replicated.

2.3 $\mathbb{P}$-Cost-efficiency

For a given probability measure $\mathbb{P} \in \mathcal{P}$, Dybvig (1988a,b) introduced the cost-efficiency problem

$$\inf_{X \in \mathcal{A}^{\mathbb{P}}_{F_0}} E_{\mathbb{Q}}[X], \tag{2.1}$$

where $\mathcal{A}^{\mathbb{P}}_{F_0}$ denotes the set of all payoffs that are ($\mathbb{P}$-)distributed with F_0.

Note that in order to render the cost-efficiency problem 2.1 well-posed, i.e., that F_0 is such that

$$\forall X \in \mathcal{A}_{F_0}, \quad E_{\mathbb{Q}}[|X|] < \infty, \tag{2.2}$$

it is sufficient to assume square-integrability of the state-price $\xi^{\mathbb{P}} := \frac{1}{\ell^{\mathbb{P}}}$. Indeed, in this case it follows from the Cauchy-Schwarz inequality that $\forall X \in \mathcal{A}_{F_0}$,

$$E_{\mathbb{Q}}[|X|] = E_{\mathbb{P}}\left[\left|\frac{X}{\ell^{\mathbb{P}}}\right|\right] \leqslant E_{\mathbb{P}}\left[|X|^2\right] E_{\mathbb{P}}\left[\left|\frac{1}{\ell^{\mathbb{P}}}\right|^2\right] < \infty.$$

Definition 2.7. A payoff X is called $\mathbb{P}-$*cost-efficient* if X solves the cost-efficiency problem under $\mathbb{P}$ for some distribution F_0.

The cost-efficiency problem (2.1) has been solved in Dybvig (1988a,b) and in Bernard et al. (2014).

Lemma 2.8. *(Dybvig (1988a,b), Bernard et al. (2014)) Assume $\ell^{\mathbb{P}}$ is continuously distributed under $\mathbb{P}$. Let F_0 be a distribution function. There is a $\mathbb{P}-a.s.$ unique optimizer to the cost-efficiency problem under the probability measure $\mathbb{P}$ given by*

$$X^* = F_0^{-1}\left(F_{\ell^{\mathbb{P}}}^{\mathbb{P}}\left(\ell^{\mathbb{P}}\right)\right)$$

that is left-continuous and non-decreasing $\mathbb{P}-a.s.$ in $\ell^{\mathbb{P}}$. It further holds that a payoff $X \in \mathcal{A}_{F_0}$ is $\mathbb{P}-$cost-efficient if and only if it is non-decreasing in the likelihood ratio, $\ell^{\mathbb{P}}$, $\mathbb{P}-$almost surely.

Proof. See Bernard et al. (2014, Corollary 2 and Proposition 2). □

In Section 3, we provide a "robust" version of the cost-efficiency problem to account for ambiguity, and in Section 5 we prove the close relationship between the robust cost-efficient payoffs and the solutions to the robust expected utility problem of Gilboa and Schmeidler (1989).

3 Robust cost-efficiency

In a situation in which there is ambiguity, i.e., when $|\mathcal{P}| > 1$, the payoff distribution of a future random payoff X changes when varying the priors $\mathbb{P} \in \mathcal{P}$, and hence it can no longer coincide with the target distribution F_0 for all priors. To accommodate ambiguity, we require that under all priors $\mathbb{P} \in \mathcal{P}$, $F_X^{\mathbb{P}}$ dominates the benchmark distribution F_0 w.r.t. some stochastic integral ordering.

Definition 3.1 (Stochastic integral ordering)**.** Let $\mathbb{F}$ be a set of measurable functions from $\mathbb{R}$ to $\mathbb{R}$. G dominates F in *integral stochastic ordering*, in notation $F \preceq_{\mathbb{F}} G$, if

$$\forall f \in \mathbb{F}, \quad \int_{\mathbb{R}} f(x)dF \leqslant \int_{\mathbb{R}} f(x)dG, \tag{3.1}$$

such that these expectations are finite.

We refer to Denuit et al. (2005) and references therein for an introduction to integral stochastic ordering. Let $\mathbb{F}_{FSD}$ denote the set of all non-decreasing functions. The corresponding stochastic integral ordering is called first order stochastic dominance (FSD).

Definition 3.2 (FSD ordering)**.** Let F and G be two distribution functions. G dominates F in the sense of *first order stochastic dominance*, in notation $F \preceq_{FSD} G$, if $F \preceq_{\mathbb{F}_{FSD}} G$.

Furthermore, let $\mathbb{F}_{SSD}$ denote the set of all non-decreasing and concave functions. the corresponding stochastic integral ordering is then labelled as second order stochastic dominance (SSD).

Definition 3.3 (SSD ordering)**.** Let F and G be two distribution functions. G dominates F in the sense of *second order stochastic dominance*, in notation $F \preceq_{SSD} G$, if $F \preceq_{\mathbb{F}_{SSD}} G$.

Lemma 3.4 (Characterization of FSD and SSD ordering)**.** *Let F and G be two distribution functions. Then $F \preceq_{FSD} G$, if and only if*

$$F^{-1}(p) \leqslant G^{-1}(p), \;\; p \in [0,1],$$

and $F \preceq_{SSD} G$, if and only if

$$\int_0^p F^{-1}(t)dt \leqslant \int_0^p G^{-1}(t)dt, \;\; p \in [0,1]. \tag{3.2}$$

Proof. These results are well-known and can be found for instance in Hanoch and Levy (1969, Theorems 1 and 2). Related results are also discussed in Denuit et al. (2005, Proposition 3.3.14), Denuit et al. (2005, Section 3.4) and Föllmer and Schied (2011, Theorem 2.57). □

Lemma 3.5. *Let $\mathbb{P}^*$ and $\mathbb{P}$ be two probability measures and X a payoff with distribution functions $F_X^{\mathbb{P}^*}$ and $F_X^{\mathbb{P}}$, respectively. It holds that $F_X^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_X^{\mathbb{P}}$ if and only if $E_{\mathbb{P}^*}[f(X)] \leqslant E_{\mathbb{P}}[f(X)]$ for all $f \in \mathbb{F}$ such that expectations are finite.*

Proof. It holds that

$$E_{\mathbb{P}}[f(X)] = \int_{\mathbb{R}} f(x) d\mathbb{P}_X = \int_{\mathbb{R}} f(x) dF_X^{\mathbb{P}}.$$

□

Definition 3.6 (Robust cost-efficiency problem)**.** Let F_0 denote a target distribution. The $\mathbb{F}-$*robust cost-efficiency problem* for F_0 is defined as

$$\inf_{X \in \mathcal{B}_{F_0}^{\mathbb{F}}} E_{\mathbb{Q}}[X], \tag{3.3}$$

in which $\mathcal{B}_{F_0}^{\mathbb{F}}$ denotes a class of admissible payoffs defined as

$$\mathcal{B}_{F_0}^{\mathbb{F}} = \left\{ X \in \mathcal{X} \,:\, \forall \mathbb{P} \in \mathcal{P},\, F_0 \preceq_{\mathbb{F}} F_X^{\mathbb{P}} \text{ and } E_{\mathbb{Q}}[X] \in \mathbb{R} \right\}.$$

We thus look for the best (i.e., cheapest) possible payoff whilst ensuring that in the worst case its distribution is still better than F_0 in the sense of some integral stochastic order (e.g., FSD or SSD), which reflects the basic idea that agents aim for protection against the worst, whilst hoping for the best. A solution to (3.3) is called a $\mathbb{F}$-*robust cost-efficient payoff*. When $\mathbb{F} = \mathbb{F}_{FSD}$, respectively $\mathbb{F} = \mathbb{F}_{SSD}$, we call such payoff FSD-robust cost-efficient, respectively SSD-robust cost-efficient.

In Theorem 3.17 we provide solutions to the robust cost-efficiency problem (3.3) under suitable conditions on the set $\mathbb{F}$. To this end we introduce the concept of regularity of $\mathbb{F}$, for which a few definitions are first needed.

3.1 Regularity of the set $\mathbb{F}$

Definition 3.7 (Composition-consistency of $\mathbb{F}$)**.** A set of measurable functions $\mathbb{F}$ is said to be *composition-consistent* if for $f, g \in \mathbb{F}$ also $f \circ g \in \mathbb{F}$.

Proposition 3.8. *The sets $\mathbb{F}_{FSD}$ and $\mathbb{F}_{SSD}$ are composition-consistent.*

Proof. The proof follows from the fact that the composition of a non-decreasing and concave (resp. non-decreasing) function is again non-decreasing and concave (resp. non decreasing). □

Lemma 3.9. *Let $\mathbb{P}, \mathbb{P}^* \in \mathcal{P}$. Let $\mathbb{F}$ be a composition-consistent set of measurable functions. Let X be a payoff and $f \in \mathbb{F}$. It holds that*

$$F_X^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_X^{\mathbb{P}} \Rightarrow F_{f(X)}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{f(X)}^{\mathbb{P}}.$$

Proof. The claim follows from Lemma 3.5 and the fact that $\mathbb{F}$ is composition-consistent. □

Definition 3.10 ($\mathbb{P}-$cost-consistency of $\mathbb{F}$)**.** Let $\mathbb{P} \in \mathcal{P}$. Let $X, Y \in \mathcal{X}$ such that X, Y are $\mathbb{P}-$cost-efficient. A set of measurable functions $\mathbb{F}$ is called $\mathbb{P}-$*cost-consistent* if $F_X^{\mathbb{P}} \preceq_{\mathbb{F}} F_Y^{\mathbb{P}}$ implies $E_{\mathbb{Q}}[X] \leqslant E_{\mathbb{Q}}[Y]$ and, additionally, $F_X^{\mathbb{P}} \neq F_Y^{\mathbb{P}}$ implies $E_{\mathbb{Q}}[X] < E_{\mathbb{Q}}[Y]$.

As the set $\mathbb{F}_{SSD}$ is contained in $\mathbb{F}_{FSD}$, the following proposition implies that $\mathbb{F}_{FSD}$ and $\mathbb{F}_{SSD}$ are $\mathbb{P}-$cost-consistent. To provide more intuition, a direct proof for the $\mathbb{P}-$cost-consistency of $\mathbb{F}_{FSD}$ can be found in Appendix B.

Proposition 3.11. *If $\mathbb{F}$ is a set of measurable functions such that $\mathbb{F}_{SSD} \subset \mathbb{F}$, then $\mathbb{F}$ is $\mathbb{P}-$cost-consistent for all $\mathbb{P} \in \mathcal{P}$.*

Proof. Following the lines of the proof of Lemma 2 in Bernard et al. (2019), one can show that $\mathbb{F}_{SSD}$ is $\mathbb{P}-$cost-consistent for all $\mathbb{P} \in \mathcal{P}$. Furthermore, $F_X^{\mathbb{P}} \preceq_{\mathbb{F}} F_Y^{\mathbb{P}}$ implies $F_X^{\mathbb{P}} \preceq_{\mathbb{F}_{SSD}} F_Y^{\mathbb{P}}$, which finishes the proof. □

To solve the robust cost-efficiency problem (3.3), we need to assume the existence of a "least favourable measure," defined via Definition 3.12 hereafter. This definition generalizes Definition 2.1 of Schied (2005), who considered the case $\mathbb{F} = \mathbb{F}_{FSD}$. Schied (2005) solved the maximization problem of robust utility under the assumption that there exists a least favourable measure.

Definition 3.12 (Least favourable measure with respect to $\mathbb{F}$)**.** Let $\mathbb{F}$ be a set of measurable functions. A measure $\mathbb{P}^* \in \mathcal{P}$ with corresponding likelihood ratio ℓ^* is called the *least favourable measure with respect to* $\mathbb{F}$ if $F_{\ell^*}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{\ell^*}^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$.

Verifying the existence of the least favourable measure as in Definition 3.12 requires the evaluation of inequalities of the form $\int f(x) dF_{\ell^*}^{\mathbb{P}^*} \leqslant \int f(x) dF_{\ell^*}^{\mathbb{P}}$ for all functions $f \in \mathbb{F}$ and all $\mathbb{P} \in \mathcal{P}$. The following propositions give an equivalent definition of the least favourable measure, involving integral inequalities with respect to the martingale measure $\mathbb{Q}$ in the important cases $\mathbb{F} = \mathbb{F}_{FSD}$ and $\mathbb{F} = \mathbb{F}_{SSD}$; the proofs are deferred to Appendix C.

Proposition 3.13 (Least favourable measure with respect to $\mathbb{F}_{FSD}$)**.** *The measure $\mathbb{P}^* \in \mathcal{P}$, with corresponding likelihood ratio ℓ^*, is the least favourable measure with respect to $\mathbb{F}_{FSD}$ if and only if ℓ^* is the convex minimal element w.r.t. $\mathbb{Q}$ in the class of likelihood ratios $\mathcal{L} := \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} : \mathbb{P} \in \mathcal{P} \right\}$, i.e.,*

$$\mathbb{E}_{\mathbb{Q}}[v(\ell^*)] = \int v(x) dF_{\ell^*}^{\mathbb{Q}} \leqslant \int v(x) dF_{\ell}^{\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}}[v(\ell)]$$

for all convex functions $v \colon \mathbb{R} \to \mathbb{R}$ and $\ell \in \mathcal{L}$.

Proposition 3.14 (Least favourable measure with respect to $\mathbb{F}_{SSD}$)**.** *The measure $\mathbb{P}^* \in \mathcal{P}$, with corresponding likelihood ratio ℓ^*, is the least favourable measure with respect to $\mathbb{F}_{SSD}$ if and only if*

$$\mathbb{E}_{\mathbb{Q}}[v(\ell^*)] = \int v(x) dF_{\ell^*}^{\mathbb{Q}} \leqslant \int v(x) dF_{\ell}^{\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}}[v(\ell)]$$

for all convex differentiable functions $v \colon \mathbb{R} \to \mathbb{R}$ with concave derivative v' and $\ell \in \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} : \mathbb{P} \in \mathcal{P} \right\}$.

Definition 3.15 (Regularity conditions for $\mathbb{F}$)**.** We say a set of functions $\mathbb{F}$ meets the *regularity conditions* if there exists a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ with respect to $\mathbb{F}$, with corresponding likelihood ratio ℓ^*, such that $x \mapsto F_{\ell^*}^{\mathbb{P}^*}(x)$ is continuous, $\mathbb{F}$ is composition-consistent, and $\mathbb{F}$ is $\mathbb{P}^*-$cost-consistent.

Note that both $\mathbb{F}_{FSD}$ and $\mathbb{F}_{SSD}$ meet the regularity conditions as soon as there exists a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ with corresponding likelihood ratio ℓ^* with respect to $\mathbb{F}_{FSD}$ (resp. $\mathbb{F}_{SSD}$) such that $x \mapsto F_{\ell^*}^{\mathbb{P}^*}(x)$ is continuous.

The next corollary follows immediately and is helpful in verifying that a set $\mathbb{F}$ meets the regularity conditions.

Corollary 3.16 (Sufficient conditions for regularity)**.** *Let $\mathbb{P}^* \in \mathcal{P}$ with corresponding likelihood ratio $\ell^* := \ell^{\mathbb{P}^*}$. Assume S_T is continuously distributed under $\mathbb{P}^*$. Let $\mathbb{F}$ be a set of measurable functions that is composition-consistent and $\mathbb{P}^*-$ cost-consistent. If $F_{S_T}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{S_T}^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$ and $\ell^* = f(S_T)$ for some strictly increasing function $f \in \mathbb{F}$, then it holds that $F_{\ell^*}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{\ell^*}^{\mathbb{P}}$ and ℓ^* is continuously distributed under $\mathbb{P}^*$, i.e., $\mathbb{F}$ meets the regularity conditions.*

3.2 Solution to the $\mathbb{F}-$robust cost-efficiency problem

We now provide our main result for this section.

Theorem 3.17 ($\mathbb{F}$-robust cost-efficient payoff)**.** *Assume a set of functions $\mathbb{F}$ meets the regularity conditions with respect to the corresponding least favourable measure $\mathbb{P}^* \in \mathcal{P}$. If $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$ then the $\mathbb{F}-$robust cost-efficiency problem has a $\mathbb{P}^*-$a.s. unique solution given by*

$$F_0^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

Proof. Denote by ℓ^* the likelihood ratio that corresponds to $\mathbb{P}^*$. Let $X^* = F_0^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right)$. As $\mathbb{F}$ meets the regularity conditions and $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$, it follows from Lemma 3.9 that $F_0 = F_{X^*}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{X^*}^{\mathbb{P}}$; hence X^* is admissible. Let $Y \in \mathcal{B}_{F_0}^{\mathbb{F}}$ and

$$Y^* = \left[F_Y^{\mathbb{P}^*}\right]^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

Then Y^* is cost-efficient and $F_{X^*}^{\mathbb{P}^*} = F_0 \preceq_{\mathbb{F}} F_Y^{\mathbb{P}^*} = F_{Y^*}^{\mathbb{P}^*}$ and because $\mathbb{F}$ is $\mathbb{P}^*-$cost-consistent, it follows that $E_{\mathbb{Q}}[X^*] \leqslant E_{\mathbb{Q}}[Y^*] \leqslant E_{\mathbb{Q}}[Y]$. Hence, every admissible payoff is more expensive than X^*. We now show uniqueness. Let $\hat{X}$ be another solution to the robust cost-efficiency problem. It holds that $F_0 \preceq_{\mathbb{F}} F_{\hat{X}}^{\mathbb{P}^*}$. If $F_{\hat{X}}^{\mathbb{P}^*} = F_0$ and $E_{\mathbb{Q}}[X^*] = E_{\mathbb{Q}}[\hat{X}]$, then $X^* = \hat{X}$, $\mathbb{P}^*-$a.s. by Lemma 2.8 because the solution X^* corresponds to the solution of the standard $\mathbb{P}^*-$cost-efficency problem, which has a unique solution. If $F_0 \neq F_{\hat{X}}^{\mathbb{P}^*}$, then $E_{\mathbb{Q}}[X^*] < E_{\mathbb{Q}}[\hat{X}]$ because $\mathbb{F}$ is cost-consistent. Hence, X^* is the unique solution to the robust cost-efficiency problem. □

Corollary 3.18 (FSD-robust cost-efficient payoff)**.** *Assume that there exists a least favourable measure $\mathbb{P}^*$ with respect to $\mathbb{F}_{FSD}$ and let ℓ^* denote the corresponding likelihood ratio. If $x \mapsto F_{\ell^*}^{\mathbb{P}^*}(x)$ is continuous., then the FSD-robust cost-efficiency problem has a $\mathbb{P}^*-$a.s. unique solution given by*

$$F_0^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

Corollary 3.19 (SSD-robust cost-efficient payoff)**.** *Assume that there exists a least favourable measure $\mathbb{P}^*$ with respect to $\mathbb{F}_{SSD}$ and let ℓ^* denote the corresponding likelihood ratio. If $x \mapsto F_{\ell^*}^{\mathbb{P}^*}(x)$ is continuous and, in addition, $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ is concave, then the SSD-robust cost-efficiency problem has a $\mathbb{P}^*-$a.s. unique solution given by*

$$F_0^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

The condition in Corollary 3.19 that the function $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ must be concave means that the target distribution F_0 is required to be lighter-tailed than the distribution $F_{\ell^*}^{\mathbb{P}^*}$. Specifically, $F_{\ell^*}^{\mathbb{P}^*}$ must dominate F_0 in the sense of transform convex order (Shaked and Shanthikumar (2007)).

Third order stochastic dominance is the stochastic integral ordering that arises from the set $\mathbb{F}_{TSD}$, containing all functions f such that $f' > 0$, $f'' \leqslant 0$ and $f''' \geqslant 0$.

Proposition 3.20. *Let $\mathbb{P}^*$ be a least favourable measure with respect to the set $\mathbb{F}_{TSD}$. The set $\mathbb{F}_{TSD}$ is composition-consistent but is in general not $\mathbb{P}^*-$cost-consistent.*

Proof. See Appendix D. □

Proposition 3.20 implies that the set $\mathbb{F}_{TSD}$ does not meet the regularity conditions, and hence Theorem 3.17 cannot be applied to find robust optimal payoffs.

Remark 3.21. Müller et al. (2017) introduced the $(1+\gamma)-$stochastic dominance order for $\gamma \in (0,1)$, which lies between FSD and SSD order. We label it by $(1+\gamma)-$SD order and denote the corresponding set of measurable functions by $\mathbb{F}_{1+\gamma}$. The set $\mathbb{F}_{1+\gamma}$ is in general not composition-consistent but it is cost-consistent in light of Proposition 3.11.

Remark 3.22. Rothschild and Stiglitz (1970) introduced concave stochastic order, which is defined via the set $\mathbb{F}_{con}$, which contains all concave (but not necessarily non-decreasing) functions. It holds that $\mathbb{F}_{SSD} \subset \mathbb{F}_{con}$ but $\mathbb{F}_{con} \not\subset \mathbb{F}_{FSD}$. The partial order $\preceq_{\mathbb{F}_{con}}$ coincides with $\preceq_{SSD}$ if we compare two payoffs with the same mean, see (Föllmer and Schied, 2011, Remark 2.63). The set $\mathbb{F}_{con}$ is cost-consistent but not composition-consistent.

Remark 3.23. Attentive readers might wonder what happens if there is another least favourable measure $\mathbb{P}^+$ with corresponding likelihood ratio ℓ^+ also satisfying the regularity condition, i.e., $F_{\ell^+}^{\mathbb{P}^+} \preceq_{\mathbb{F}} F_{\ell^+}^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$. Then $X^+ = F_0^{-1}\left(F_{\ell^{\mathbb{P}^+}}^{\mathbb{P}^+}(\ell^+)\right)$ also solves the $\mathbb{F}-$robust cost-efficiency problem. The proof above shows that $X^+ = X^*$, $\mathbb{P}^*-$a.s. and hence also $\mathbb{P}^+-$a.s. because the two measures are equivalent to each other. We provide another view to show that $X^+ = X^*$ $\mathbb{P}^+-$ a.s. Note that

$$F_0 = F_{X^*}^{\mathbb{P}^*} \preceq_{\mathbb{F}} F_{X^*}^{\mathbb{P}^+} \text{ and } F_0 = F_{X^+}^{\mathbb{P}^+} \preceq_{\mathbb{F}} F_{X^+}^{\mathbb{P}^*}.$$

As $\mathbb{F}$ is $\mathbb{P}^*-$cost-consistent, it follows that $E_{\mathbb{Q}}[X^*] \leqslant E_{\mathbb{Q}}[X^+]$. If $E_{\mathbb{Q}}[X^*] < E_{\mathbb{Q}}[X^+]$, we are done. Otherwise, applying cost-consistency again, it must hold that $F_{X^*}^{\mathbb{P}^*} = F_{X^+}^{\mathbb{P}^*}$. The payoff X^* is $\mathbb{P}^*-$cost-efficient and has the same price and distribution as X^+ under $\mathbb{P}^*$. Hence, by Lemma 2.8 it holds that $X^+ = X^*$, $\mathbb{P}^*-$a.s.

We now discuss some examples to illustrate Theorem 3.17 in a log-normal market setting with uncertainty on the drift and volatility (Sections 3.3.1 and Section 3.3.2), and in a more general market setting (Section 3.4).

3.3 Robust cost-efficient payoffs in lognormal markets

We assume that the observed call prices reveal that there exists a measure $\mathbb{Q}$ such that under $\mathbb{Q}$, S_T has a log-normal distribution with parameters $(r - \frac{s^2}{2})T$ and s^2T. Under $\mathbb{Q}$, S_T is log-normally distributed with density

$$f^r(x) = \frac{S_0}{xs\sqrt{T}}\,\varphi\left(\frac{\ln\left(\frac{x}{S_0}\right) - (\mu - \frac{s^2}{2})T}{s\sqrt{T}}\right),$$

where φ denotes the density of the standard normal distribution. For simplicity, we assume that the price of the stock today is $S_0 = 1$.

3.3.1 Drift uncertainty: FSD robust cost-efficient payoff

The real-world distribution of S_T is assumed to be log-normal with parameters $(\mu - \frac{s^2}{2})T$ and s^2T, but there is uncertainty about the precise level of the drift parameter μ. In particular, the agent only expects the true drift parameter μ to lie in the interval $\mathcal{D}^{\mu_1} = \{\mu \in \mathbb{R} : r < \mu_1 \leqslant \mu < \infty\}$ and thus considers $\mathcal{P} = (\mathbb{P}^{\mu_1})_{\mu\in\mathcal{D}^{\mu_1}}$ as the set of all plausible probability measures on $(\Omega, \mathcal{F})$. Under $\mathbb{P}^{\mu}$, S_T is log-normal with density

$$f^{\mu}(x) = \frac{S_0}{xs\sqrt{T}}\,\varphi\left(\frac{\ln\left(\frac{x}{S_0}\right) - (\mu - \frac{s^2}{2})T}{s\sqrt{T}}\right),$$

It follows that

$$S_T^{\mathbb{P}^{\mu_1}} \leqslant S_T^{\mathbb{P}^{\mu}},\ \ a.s.,$$

which implies that $S_T^{\mathbb{P}^*} \preceq_{FSD} S_T^{\mathbb{P}^{\mu}}$ for all $\mu \geqslant \mu_1$, where $\mathbb{P}^* := \mathbb{P}^{\mu_1}$. Let $\ell^{\mathbb{P}^{\mu_1}} = \frac{f^{\mu_1}(S_T)}{f^{r,s}(S_T)}$. A straightforward computation shows that

$$\ell^{\mathbb{P}^{\mu_1}} = \left(\frac{S_T}{S_0}\right)^{\frac{\mu_1 - r}{s^2}} \exp\left(-\frac{\left(\left(\mu_1 - \frac{s^2}{2}\right)^2 + \left(r - \frac{s^2}{2}\right)^2\right)T}{2s^2}\right). \tag{3.4}$$

Hence, $\ell^{\mathbb{P}^{\mu_1}}$ is a strictly increasing function of S_T as $\mu_1 > r$. By Corollary 3.16, the set $\mathbb{F}_{FSD}$ meets the regularity conditions with least favourable measure $\mathbb{P}^*$ and corresponding likelihood ratio ℓ^*. Theorem 3.17 shows that the robust cost-efficient payoff for a distribution function F_0 with finite cost is given by

$$X^* = F_0^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right) = F_0^{-1}\left(F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T)\right).$$

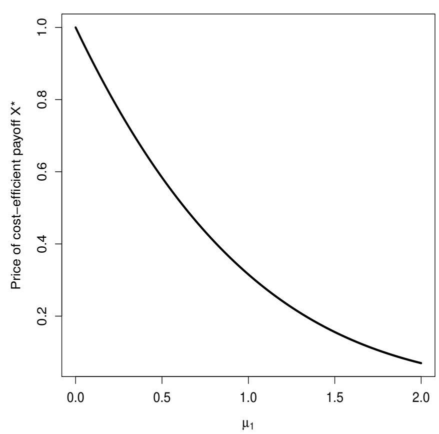
The second equality follows from the increasingness of $\ell^{\mathbb{P}^{\mu_1}}$ in S_T. The agent thus chooses the optimal payoff as if he believes that the worst-case plausible value for the drift parameter μ, i.e., μ_1, will materialize. This finding is consistent with the results obtained Schied (2005, Section 3.1) on the impact of drift uncertainty on optimal payoff choice in a time-dynamic Black-Scholes setting.

We consider next the exponential distribution for the target distribution F_0, i.e., $F_0(x) = 1 - e^{-x}$. Panel A of Figure 1 displays the price of the robust cost-efficient payoff for varying levels of the parameter μ_1, which describes the ambiguity the agent faces. The higher μ_1, the smaller the set $\mathcal{D}^{\mu_1}$ and the cheaper X^*.

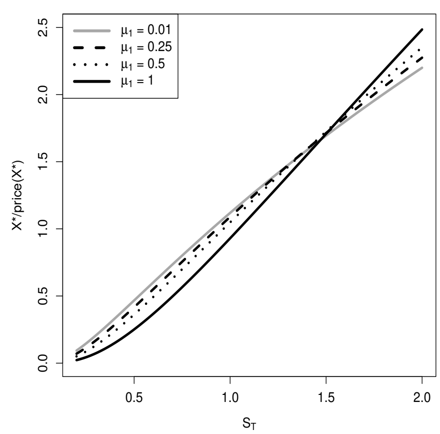
In panel B of Figure 1 we display, for several values of μ_1, the robust cost-efficient payoff normalized for its initial price as a function of S_T), i.e., we display the curve

$$S_T \mapsto \frac{F_0^{-1}\left(F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T)\right)}{e^{-rT}E_{\mathbb{Q}}\left[F_0^{-1}\left(F_{S_T}^{\mathbb{P}^{\mu_1}}(S_T)\right)\right]}.$$

We observe that the curve is flatter when μ_1 is smaller, i.e., increasing ambiguity translates into more payoffs that reflect a higher degree of conservatism.



Panel A



Panel B

Figure 1: We use the parameters $S_0 = 1$, $r = 0$, $T = 1$ and $s = 0.9$. The reference distribution is the exponential distribution $F_0(x) = 1 - e^{-x}$. Panel A: Price of the cost-efficient payoff X^* depending on the value of the ambiguity parameter μ_1. Panel B: Cost-efficient payoff per unit of investment for various values of μ_1.

3.3.2 Drift and volatility uncertainty: SSD robust cost-efficient payoff

The real-world distribution of S_T is assumed to be log-normal with parameters $(\mu - \frac{\sigma^2}{2})T$ and $\sigma^2 T$, but now the agent faces uncertainty about the precise level of the parameters μ and σ. In particular, the agent only expects the true parameters to lie within the cube

$$\mathcal{D}^{\mu_1,s} = \left\{(\mu,\sigma) \subset \mathbb{R}^2 \,:\, r < \mu_1 \leqslant \mu \leqslant \mu_2,\, 0 < \sigma_1 \leqslant \sigma \leqslant s\right\}.$$

In this regard, note that while $r < \mu_1 \leqslant \mu \leqslant \mu_2$ is a natural assumption, there is some empirical evidence for the hypothesis that $\sigma \leqslant s$; see Table 1 in Christensen and Prabhala (1998) and Christensen and Hansen (2002).

Remark 3.24. In contrast to the dynamic Black-Scholes model, in which the stock price S_T is also log-normally distributed, we work in a static market setting. In a dynamic Black-Scholes framework where continuous trading is allowed at zero transaction cost, the absence of arbitrage opportunities implies that the volatility of the stock does not change when moving from the real-world measure to the risk-neutral measure, i.e., there does not exist uncertainty about the volatility in a dynamic Black-Scholes model. Here, however, we do not assume dynamic trading. Hence, even when call option prices reflect a risk neutral distribution for S_T that is log-normal distributed, the agent may have a view on the real-world distribution that is different from a log-normal and, in particular, may be unsure about the exact values for drift and volatility.

Proposition 3.25 (SSD robust cost-efficient payoff)**.** *If $\frac{\mu_1 - r}{s^2} \in (0,1]$ then it holds that $F_{S_T}^{\mathbb{P}^{\mu_1,s}} \preceq_{SSD} F_{S_T}^{\mathbb{P}^{\mu,\sigma}}$, $(\mu,\sigma) \in \mathcal{D}$ and the set $\mathbb{F}_{SSD}$ meets the regularity conditions with least favourable measure $\mathbb{P}^* = \mathbb{P}^{\mu_1,s}$ and corresponding likelihood ratio $\ell^* = \ell^{\mathbb{P}^{\mu_1,s}}$. The SSD robust cost-efficient payoff for a distribution function F_0 such that $F_0^{-1} \circ F_{\ell^*}^{\mathbb{P}^*}$ is concave is then given by*

$$X^* := F_0^{-1}\left(F_{S_T}^{\mathbb{P}^{\mu_1,s}}(S_T)\right). \tag{3.5}$$

Proof. For a log-normal distribution function F with parameters M and V, it holds that

$$\int_0^q F^{-1}(p)dp = \frac{e^{M+\frac{V}{2}}}{q}\ \Phi\left(\Phi^{-1}(q) - \sqrt{V}\right), \quad q \in (0,1),$$

where Φ denotes the distribution function of a standard normal payoff. It follows that

$$\begin{aligned}\int_0^q \left[F_{S_T}^{\mathbb{P}^{\mu_1,s}}\right]^{-1}(p)dp &= \frac{e^{\mu_1 T}}{q}\ \Phi\left(\Phi^{-1}(q) - s\sqrt{T}\right)\\ &\leqslant \frac{e^{\mu T}}{q}\ \Phi\left(\Phi^{-1}(q) - \sigma\sqrt{T}\right),\ \ q \in (0,1),\ \ \mu_1 \leqslant \mu,\ \ \sigma \leqslant s.\end{aligned}$$

Hence, $F_{S_T}^{\mathbb{P}^{\mu_1,s}} \preceq_{SSD} F_{S_T}^{\mathbb{P}^{\mu,\sigma}}$, $(\mu,\sigma) \in \mathcal{D}$. As in Section 3.3.1, define $\ell^{\mathbb{P}^{\mu_1,s}}$, $f^{\mu_1,s}$ and $f^{r,s}$. The likelihood ratio $\ell^{\mathbb{P}^{\mu_1,s}}$ is strictly increasing and concave in S_T, if $\frac{\mu_1-r}{s^2} \in (0,1]$; compare with Equation (3.4). By Corollary 3.16, the set $\mathbb{F}_{SSD}$ meets the regularity condition for the measure $\mathbb{P}^* = \mathbb{P}^{\mu_1,s}$ with likelihood ratio $\ell^* = \ell^{\mathbb{P}^{\mu_1,s}}$. As in Section 3.3.1, a little calculation and Theorem 3.17 show that the robust cost-efficient payoff for a distribution function F_0 with finite cost is given by Equation (3.5). □

Remark 3.26. Let $\gamma = \frac{\mu_1-r}{s^2}$. There are economically reasonable environments such that $\gamma \in (0,1]$. For example, if $s \in [0.2,\infty)$ and $(\mu-r) \in (0,0.04]$ or if $s \in [0.35,\infty)$ and $(\mu-r) \in (0,0.1]$, it follows that $\gamma \in (0,1]$.

Remark 3.27. Note that the quantile functions $\left[F_{S_T}^{\mathbb{P}^{\mu_1,s}}\right]^{-1}$ and $\left[F_{S_T}^{\mathbb{P}^{\mu_1,\sigma}}\right]^{-1}$ are crossing and therefore the SSD-robust optimal payoff does not solve the FSD-robust cost-efficiency problem.

3.4 Robust cost-efficient payoffs in general markets using Esscher transform

Inspired by Corcuera et al. (2009), let $S_0 = 1$ and $s > 0$ and Z be a payoff with mean zero and variance one. Under $\mathbb{Q}$, assume that Z has density $f_Z^{\mathbb{Q}}(x) > 0$, $x \in \mathbb{R}$ and model the future stock price at date T by

$$S_T = S_0 e^{(r+\omega)T + s\sqrt{T}Z},$$

where r is the risk-free rate and $\omega \in \mathbb{R}$ is a mean correcting term, i.e., ω is chosen such that

$$e^{-rT} E_{\mathbb{Q}}\left[S_T\right] = S_0. \tag{3.6}$$

The density of $X = (r+\omega)T + s\sqrt{T}Z$ under $\mathbb{Q}$ is

$$f_X^{\mathbb{Q}}(x) = \frac{1}{s\sqrt{T}} f_Z^{\mathbb{Q}}\left(\frac{x-(r+\omega)T}{s\sqrt{T}}\right).$$

The corresponding density of S_T under $\mathbb{Q}$ is denoted by $f_{S_T}^{\mathbb{Q}}$, and it holds that

$$f_{S_T}^{\mathbb{Q}}(x) = f_X^{\mathbb{Q}}(\log(x))\frac{1}{x}.$$

The density $f_{S_T}^{\mathbb{Q}}$ can be estimated from call prices in a nonparametric manner, see Aït-Sahalia and Lo (1998).

Let $h^* > 0$ and $\mathcal{H} \subset [h^*,\infty)$ be a set containing h^* such that $E_{\mathbb{Q}}\left[S_T^h\right]$ exists for all $h \in \mathcal{H}$. Define a family of probability measures $\mathcal{P} = \left(\mathbb{P}^h\right)_{h\in\mathcal{H}}$ as follows: $\mathbb{P}^h$ is a measure such that X has density $f_X^{\mathbb{P}^h}$ under $\mathbb{P}^h$, where $f_X^{\mathbb{P}^h}$ is obtained from $f_X^{\mathbb{Q}}$ by applying the Esscher transform. In particular, we define $\mathbb{P}^h$ such that

$$f_X^{\mathbb{P}^h}(x) = \frac{e^{hx} f_X^{\mathbb{Q}}(x)}{\int_{\mathbb{R}} e^{hy} f_X^{\mathbb{Q}}(y)dy} = \frac{e^{hx} f_X^{\mathbb{Q}}(x)}{E_{\mathbb{Q}}\left[S_T^h\right]}.$$

It follows that

$$f_{S_T}^{\mathbb{P}^h}(x) = \frac{x^h}{x} \frac{f_X^{\mathbb{Q}}(\log(x))}{E_{\mathbb{Q}}\left[S_T^h\right]}. \tag{3.7}$$

The density $f_{S_T}^{\mathbb{P}^{h^*}}$ crosses $f_{S_T}^{\mathbb{P}^h}$ only once from above for $h^* < h$; hence, by Denuit et al. (2005, Property 3.3.32), it follows that

$$F_{S_T}^{\mathbb{P}^{h^*}} \preceq_{FSD} F_{S_T}^{\mathbb{P}^h}, \;\; h \in \mathcal{H}. \tag{3.8}$$

For the likelihood ratio it holds

$$\ell^{\mathbb{P}^{h^*}} = \frac{f_{S_T}^{\mathbb{P}^{h^*}}(S_T)}{f_{S_T}^{\mathbb{Q}}(S_T)} = \frac{S_T^{h^*}}{E_{\mathbb{Q}}\left[S_T^{h^*}\right]}, \tag{3.9}$$

which is strictly increasing in S_T as $h^* > 0$. We can apply Corollary 3.16 to show that the set $\mathbb{F}_{SSD}$ meets the regularity conditions with least favourable measure $\mathbb{P}^* = \mathbb{P}^{h^*}$ and corresponding likelihood ratio $\ell^* = \ell^{\mathbb{P}^{h^*}}$ and use Theorem 3.17 to compute the cost-efficient payoff of a distribution function F_0.

Equation (3.8) implies that $F_{S_T}^{\mathbb{P}^{h^*}} \preceq_{SSD} F_{S_T}^{\mathbb{P}^h}$ for all $h \in \mathcal{H}$, and if $h^* \in (0,1]$ the likelihood ratio is a concave function of S_T, which implies the additional concavity property needed, e.g., in Corollary 3.19 for the set $\mathbb{F}_{SSD}$.

4 Robust portfolio selection

Gilboa and Schmeidler (1989) provide axioms that justify a maxmin expected utility framework to make decisions when there is ambiguity on the probability measure $\mathbb{P}$, i.e., when $\mathcal{P}$ is not a singleton. In this framework, Schied (2005) shows that when a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ with respect to FSD order (i.e, the stochastic integral order induced by the set $\mathbb{F}_{FSD}$ as defined in Section 3) exists, an optimal portfolio can be derived. In this section, we extend the work of Schied (2005) to more general robust preferences (i.e., in a non-expected utility setting) and under the existence of a least favourable measure $\mathbb{P}^*$ with respect to a general stochastic integral order induced by some set $\mathbb{F}$.

4.1 Robust Preferences

A *preference* is a functional W from the set of payoffs $\mathcal{X}$ to the real line. An agent with preference W prefers the payoff Y to X if $W(X) \leqslant W(Y)$. In general, W may depend on the different measures $\mathbb{P} \in \mathcal{P}$ in a complicated way. In what follows, we denote a preference that depends solely on some $\mathbb{P} \in \mathcal{P}$ by $W_{\mathbb{P}}$.

Definition 4.1. Let $(W_{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}}$ be a family of preferences. The preference $W_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$ is called $\mathbb{P}-$*law invariant* if $F_X^{\mathbb{P}} = F_Y^{\mathbb{P}}$ implies that $W_{\mathbb{P}}(X) = W_{\mathbb{P}}(Y)$. The family $(W_{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}}$ is called *law invariant* if each individual preference $W_{\mathbb{P}}$ is $\mathbb{P}-$law invariant.

Remark 4.2. If a family of preferences $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is law invariant, then for each $\mathbb{P}\in\mathcal{P}$, $W_{\mathbb{P}}$ does not depend on null sets, i.e., $W_{\mathbb{P}}(X) = W_{\mathbb{P}}(Y)$ if $X = Y$ $\mathbb{P}-$a.s. In the remainder of the article, we assume that $W_{\mathbb{P}}$ does not depend on null sets.

Example 4.3. A typical example of a $\mathbb{P}$-law invariant preference is $W_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}[u(X)]$ for some increasing utility function u. In this case, $\inf\limits_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X)$ amounts to the worst-case expected utility, commonly called *robust utility*, which was introduced in Gilboa and Schmeidler (1989).

Definition 4.4. Let $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of preferences. We consider the *robust maximization problem*

$$\max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X), \tag{4.1}$$

and the *maximization problem under a single measure* $\mathbb{P}\in\mathcal{P}$

$$\max_{X\in\mathcal{Y}^{\mathbb{P}}_{x_0}} W_{\mathbb{P}}(X) \tag{4.2}$$

where[1]

$$\mathcal{Y}_{x_0} = \{X\in\mathcal{X} : \forall\mathbb{P}\in\mathcal{P},\, W_{\mathbb{P}}[X]<\infty,\, \exists M\in\mathbb{R}, X\geq M\, a.s.,\, E_{\mathbb{Q}}[X]\leqslant x_0\} \tag{4.3}$$

and

$$\mathcal{Y}^{\mathbb{P}}_{x_0} = \{X\in\mathcal{X} : W_{\mathbb{P}}[X]<\infty,\, \exists M\in\mathbb{R}, X\geq M\, a.s.,\, E_{\mathbb{Q}}[X]\leqslant x_0\}.$$

If u is a utility function and $W_{\mathbb{P}}(.) = E_{\mathbb{P}}(u(.))$, then we also write $\mathcal{Y}_{x_0,u}$ and $\mathcal{Y}^{\mathbb{P}}_{x_0,u}$, respectively.

Remark 4.5. Note that by Jensen's inequality it holds for concave u that $E_{\mathbb{P}}[u(X)] < \infty$ if $E_{\mathbb{P}}[|X|]<\infty$. Thus, assuming $W_{\mathbb{P}}[X]<\infty$ is no restriction if the expectation of X under $\mathbb{P}$ is finite and $W_{\mathbb{P}}$ is an expected utility.

In Theorems 4.12 and 4.16 we will show that under some conditions the robust maximization problem 4.1 can be reduced to a maximization problem under a single measure $\mathbb{P}^*$. In this regard, we introduce the concept of $\mathbb{F}$-*family consistency* of a family of preferences.

Definition 4.6. Let $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of preferences. Let $\mathbb{F}$ be a set of functions and let $\mathcal{Y}$ be a set of random variables. $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is called $\mathbb{F}$-*family consistent on* $\mathcal{Y}$ *with respect to* $\mathbb{P}^*\in\mathcal{P}$ if for $Y\in\mathcal{Y}$ the inequality

$$F^{\mathbb{P}^*}_Y \preceq_{\mathbb{F}} F^{\mathbb{P}}_Y,\ \ \mathbb{P}\in\mathcal{P}$$

implies

$$W_{\mathbb{P}^*}(Y)\leqslant W_{\mathbb{P}}(Y),\ \ \mathbb{P}\in\mathcal{P}.$$

[1] This set of admissible strategies is standard in the literature on optimal portfolio selection, in particular on expected utility maximization. It ensures, e.g., that the doubling strategies are not admissible by allowing for a maximum credit line.

$\mathbb{F}-$family consistency of $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ with respect to some $\mathbb{P}^* \in \mathcal{P}$ has the following interpretation: If a measure $\mathbb{P}^*$ yields the most pessimistic view on any payoff Y w.r.t. the stochastic order induced by some set $\mathbb{F}$, then the preference under that measure is the lowest as well. In what follows, instead of saying that $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{F}_{FSD}$-family or $\mathbb{F}_{SSD}$-family consistent, respectively, we will say that they are *FSD-family consistent* and *SSD-family consistent* preferences.

Example 4.7. We consider an agent who considers a family of preferences $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ in which $W_{\mathbb{P}}$ only depends on the law of a payoff. To model such a setting, let $\mathcal{D}$ be the set of cumulative distribution functions. For each $\mathbb{P} \in \mathcal{P}$, let $w_{\mathbb{P}} : \mathcal{D} \to \mathbb{R}$ be a functional respecting FSD-order, i.e., for each $F, G \in \mathcal{D}$ such that $F \preceq_{FSD} G$ it holds that

$$w_{\mathbb{P}}(F) \leqslant w_{\mathbb{P}}(G). \tag{4.4}$$

We further assume that there is a least favourable measure $\mathbb{P}^* \in \mathcal{P}$ such that

$$w_{\mathbb{P}^*}(F) \leqslant w_{\mathbb{P}}(F), \quad F \in \mathcal{D}, \tag{4.5}$$

and we define

$$W_{\mathbb{P}}(X) = w_{\mathbb{P}}\left(F_X^{\mathbb{P}}\right).$$

Equation (4.5) then means that any payoff X is less preferred under $\mathbb{P}^*$ than under any other subjective measure $\mathbb{P}$. The family $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is FSD-family consistent because for any random variable Y as in Definition 4.6 it holds that

$$W_{\mathbb{P}^*}(Y) = w_{\mathbb{P}^*}\left(F_Y^{\mathbb{P}^*}\right) \leqslant w_{\mathbb{P}}\left(F_Y^{\mathbb{P}^*}\right) \leqslant w_{\mathbb{P}}\left(F_Y^{\mathbb{P}}\right) = W_{\mathbb{P}}(Y).$$

A similar assertion can be made if all $w_{\mathbb{P}}$ respect SSD or, more generally, some integral stochastic order. We provide some more concrete examples that fit into this setting.

i) Let $(u_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of non-decreasing and continuous utility functions $u_{\mathbb{P}} : \mathbb{R} \to \mathbb{R}$ such that $u_{\mathbb{P}^*} \leqslant u_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$. Define $w_{\mathbb{P}}$ by

$$w_{\mathbb{P}}(F) = \int_{\mathbb{R}} u_{\mathbb{P}}(x)dF, \quad F \in \mathcal{D},$$

Equations (4.4) and (4.5) are then satisfied. If each $u_{\mathbb{P}}$ is additionally concave, each $w_{\mathbb{P}}$ also respects SSD-order, i.e., for $F, G \in \mathcal{D}$ such that $F \preceq_{FSD} G$ it holds that

$$w_{\mathbb{P}}(F) \leqslant w_{\mathbb{P}}(G). \tag{4.6}$$

In particular, the exponential utility family

$$u_{\mathbb{P}}(x) = \begin{cases} \frac{1-e^{-\lambda_{\mathbb{P}} x}}{\lambda_{\mathbb{P}}} & , \lambda_{\mathbb{P}} \neq 0 \\ x & , \lambda_{\mathbb{P}} = 0, \end{cases}$$

where $\lambda_{\mathbb{P}} \in \mathbb{R}$ such that $\lambda_{\mathbb{P}^*} \geqslant \lambda_{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$, satisfies $u_{\mathbb{P}^*} \leqslant u_{\mathbb{P}}$. Hence, $W_{\mathbb{P}^*}$ corresponds to the most conservative preference.

ii) In the context of Yaari's Dual Theory, let $(\phi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ be a family of non-decreasing absolutely continuous distortion functions $\phi_{\mathbb{P}} : [0,1] \to [0,1]$ with $\phi_{\mathbb{P}}(0) = 0$ and $\phi_{\mathbb{P}}(1) = 1$ such that $\phi_{\mathbb{P}^*} \leqslant \phi_{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$. Let

$$w_{\mathbb{P}}(F) = -\int_{-\infty}^{0} 1 - \phi_{\mathbb{P}}\left(1 - F(x)\right) dx + \int_{0}^{\infty} \phi_{\mathbb{P}}\left(1 - F(x)\right) dx, \quad F \in \mathcal{D}.$$

Equations (4.4) and (4.5) are then satisfied. Again, $W_{\mathbb{P}^*}$ corresponds to the most conservative preference.

The following lemma follows immediately. In particular, if a family of preferences is SSD-family consistent, then it is also FSD-family consistent.

Lemma 4.8. *Let $\mathcal{Y}$ be a set of random variables. If $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{F}$-family consistent on $\mathcal{Y}$ and $\mathbb{F} \subset \mathbb{G}$, then $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{G}$-family consistent on $\mathcal{Y}$.*

We recall here the definition of a generalized utility function from (Bernard et al., 2015, Definition 3), as it will be useful in the next section to draw the connection between robust preferences and the maxmin expected utility setting of Gilboa and Schmeidler (1989).

Definition 4.9. (Generalized utility function). Let $\mathcal{I} = (a,b) \subset \overline{\mathbb{R}}$ be a non-empty open interval. A function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is called a *generalized utility function* if

$$u(x) = \begin{cases} u_0(x) & \text{if } x \in (a,b) \\ -\infty & \text{if } x < a \\ \lim\limits_{x \searrow a} u_0(x) & \text{if } x = a \\ \lim\limits_{x \nearrow b} u_0(x) & \text{if } x \geqslant b, \end{cases}$$

where $u_0 : (a,b) \to \mathbb{R}$ is strictly increasing and concave on (a,b). Note that the function u restricted on (a,∞) is concave, continuous and only takes (finite) real values. The set of generalized utility functions with respect to the interval $\mathcal{I}$ is denoted by $\tilde{\mathcal{U}}_{\mathcal{I}}$. Furthermore, we denote by u_0' the left derivative of u_0 on (a,b) and define u' on $\mathbb{R}$ as follows.

$$u'(x) = \begin{cases} u_0'(x) & \text{if } x \in (a,b) \\ +\infty & \text{if } x < a \\ \lim\limits_{x \searrow a} u_0'(x) & \text{if } x = a \\ \lim\limits_{x \nearrow b} u_0'(x) & \text{if } x = b \\ 0 & \text{if } x > b. \end{cases}$$

Next we show that a preference induced by a single generalized utility function is SSD-family consistent (and thus also FSD-family consistent).

Lemma 4.10. *Let $\mathcal{I} \subset \overline{\mathbb{R}}$ be a non-empty interval. Let $u \in \tilde{\mathcal{U}}_{\mathcal{I}}$ be a generalized utility function. Define $W_{\mathbb{P}}(.) = E_{\mathbb{P}}[u(.)]$ for each $\mathbb{P} \in \mathcal{P}$. Let $\mathcal{Y}_{x_0,u}$ as in Definition 4.4. Then, $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is FSD- and SSD-family consistent on $\mathcal{Y}_{x_0,u}$.*

Remark 4.11. When $a = -\infty$, the proof of the lemma is almost trivial, as $u(x)$ is then finite for all $x \in \mathbb{R}$. But a generalized utility function could also be a logarithmic function or a square-root, which is equal to $-\infty$ on $(-\infty, a]$ or $(-\infty, a)$ for $a = 0$, which makes it necessary to distinguish three cases in the proof. The full proof is relegated to Appendix E.

4.2 Optimal portfolio for robust preferences

Under the assumption of a least favourable measure $\mathbb{P}^*$ with respect to FSD-order, Schied (2005) showed that in order to solve[2] the robust maximization problem (4.1) for preferences $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$, $W_{\mathbb{P}}(x) = \mathbb{E}_{\mathbb{P}}[u(X)]$, it actually suffices to solve the single measure maximization problem (4.2). The following theorem generalizes this result beyond the expected utility setting to arbitrary $\mathbb{P}$-law invariant preferences $W_{\mathbb{P}}$.

Theorem 4.12. *Assume the set $\mathbb{F}_{FSD}$ meets the regularity condition with respect to a least favourable measure $\mathbb{P}^*$. Assume that $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is a law invariant and FSD consistent family of preferences on $\mathcal{Y}_{x_0}$ with respect $\mathbb{P}^*$. Assume the maximization problem (4.2) under $\mathbb{P}^*$ has a solution $\tilde{X} \in \mathcal{Y}_{x_0}$. Then it holds that*

$$\max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X).$$

Proof. Let h be a non-decreasing function such that $h(\ell^*) \in \mathcal{Y}_{x_0}$. Then it holds by the FSD-family consistency and Lemma 3.9

$$W_{\mathbb{P}^*}(h(\ell^*)) \leqslant \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(h(\ell^*)). \tag{4.7}$$

Let

$$X^* = \left[F_{\tilde{X}}^{\mathbb{P}^*}\right]^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

Then X^* is cost-efficient and thus $E_{\mathbb{Q}}[X^*] \leqslant E_{\mathbb{Q}}[\tilde{X}]$ and $F_{\tilde{X}}^{\mathbb{P}^*} = F_{X^*}^{\mathbb{P}^*}$: hence, by the law invariance of $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$, it holds that $X^* \in \mathcal{Y}_{x_0}$. It further holds that X^* is a non-decreasing function of ℓ^*. It follows by Equation (4.7) that

$$\begin{aligned}
\max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X) &= W_{\mathbb{P}^*}(\tilde{X}) = W_{\mathbb{P}^*}(X^*) \\
&\leqslant \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X^*) \\
&\leqslant \max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) \\
&\leqslant \max_{X\in\mathcal{Y}_{x_0}} W_{\mathbb{P}^*}(X) \\
&\leqslant \max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X),
\end{aligned}$$

where the last inequality follows by $\mathcal{Y}_{x_0} \subset \mathcal{Y}_{x_0}^{\mathbb{P}^*}$. □

[2]Note that Schied (2005) allows for dynamic trading, whereas our setting is static.

From Theorem 4.12, it follows immediately that solutions to many robust preference maximization problems, such as the maxmin expected utility problem, reduce to an optimization under a single probability measure when the set $\mathbb{F}_{FSD}$ meets the regularity condition. The following example illustrates this consequence.

Example 4.13. Assume the set $\mathbb{F}_{FSD}$ meets the regularity condition with least favourable measure $\mathbb{P}^* \in \mathcal{P}$.

1. Assuming a solution exists, for any increasing utility function $u\colon (a,b) \to \mathbb{R} \cup \{-\infty\}$ where $(a,b) \subset \mathbb{R}$, it holds that

$$\max_X \left\{ \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[u(X)] \right\} = \max_X \left\{ \mathbb{E}_{\mathbb{P}^*}[u(X)] \right\}.$$

2. In the context of Yaari's Dual Theory, assuming a solution exists, for every increasing distortion function $\omega : [0,1] \to [0,1]$, it holds that

$$\max_X \min_{\mathbb{P}\in\mathcal{P}} \int_0^\infty \omega\big(\mathbb{P}(X > x)\big)\ dx = \max_X \int_0^\infty \omega\big(\mathbb{P}^*(X > x)\big)\ dx.$$

3. In the context of Cumulative Prospect Theory the objective function is given by

$$\begin{aligned} W_{\mathbb{P}}(X) := &\int_0^\infty \omega_+\big(\mathbb{P}(u_+((X-Y)^+) > x\big)\ dx \\ &- \int_0^\infty \omega_-\big(\mathbb{P}(u_-((X-Y)^-) > x\big)\ dx \end{aligned}$$

for $X \in \mathcal{X}$, where u_+, u_- are utility functions, ω_+, ω_- are distortion functions and Y is a deterministic reference point. Assuming a solution exists, it holds that

$$\max_X \min_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \max_X W_{\mathbb{P}^*}(X).$$

In the following two corollaries we show that the solution of the single or robust maximization problem is cost-efficient if it is unique.

Corollary 4.14. *Let $\mathbb{P} \in \mathcal{P}$ with corresponding likelihood ratio $\ell^{\mathbb{P}}$. Assume $W_{\mathbb{P}}$ is $\mathbb{P}-$law invariant and $\tilde{X}$ is a $\mathbb{P}-$a.s. unique solution to the maximization problem (4.2) under $\mathbb{P}$. Then $\tilde{X}$ is $\mathbb{P}-$cost-efficient.*

Proof. Let

$$X^* = \left[F_{\tilde{X}}^{\mathbb{P}^*}\right]^{-1} \left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

Then $X^* \in \mathcal{Y}_{x_0}^{\mathbb{P}}$, see proof of Theorem 4.12. It holds by law invariance that

$$\max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}}} W_{\mathbb{P}}(X) = W_{\mathbb{P}}(\tilde{X}) = W_{\mathbb{P}}(X^*).$$

As $\tilde{X}$ is the unique solution, it must hold $\tilde{X} = X^*$, $\mathbb{P} - a.s.$, and thus $\tilde{X}$ is cost-efficient. ☐

Corollary 4.15. *Assume that the set $\mathbb{F}_{FSD}$ meets the regularity conditions with least favourable measure $\mathbb{P}^* \in \mathcal{P}$. Assume that the robust maximization problem (4.1) has a unique solution $\tilde{X}$ and that $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is law invariant and FSD-family consistent on $\mathcal{Y}_{x_0}$ with respect to $\mathbb{P}^*$. Then $\tilde{X}$ is $\mathbb{P}^*-$cost-efficient.*

Proof. The proof is similar to the one for Corollary 4.14: let

$$X^* = \left[F_{\tilde{X}}^{\mathbb{P}^*}\right]^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right).$$

It holds by law invariance that

$$\max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(\tilde{X}) = \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X^*).$$

□

In contrast to Theorem 4.12, in the following theorem, the $W_{\mathbb{P}}$ do not need to be law-invariant. Moreover, the preferences also do not need to be increasing, i.e., $X \leqslant Y$ does not need to imply $W_{\mathbb{P}}(X) \leqslant W_{\mathbb{P}}(Y)$ as long as the solution is expressible as a certain function of the likelihood ratio ℓ^*.

Theorem 4.16. *Assume that a set of functions $\mathbb{F}$ meets the regularity conditions with least favourable measure $\mathbb{P}^*$. Assume that the maximization problem (4.2) under $\mathbb{P}^*$ has a solution $\tilde{X} \in \mathcal{Y}_{x_0}$, which can $\mathbb{P}^*-$a.s. be expressed as $f(\ell^*)$ for some $f \in \mathbb{F}$. Further, assume that $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is $\mathbb{F}$-family consistent on $\mathcal{Y}_{x_0}$ with respect to $\mathbb{P}^*$. Then it holds that*

$$\max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) = \max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X).$$

Proof. Let $h \in \mathbb{F}$ such that $h(\ell^*) \in \mathcal{Y}_{x_0}$. Then, Equation (4.7) holds true by the $\mathbb{F}$-family consistency and Lemma 3.9. By assumption, $\tilde{X} = f(\ell^*)$, $\mathbb{P}^*-$a.s. for some $f \in \mathbb{F}$. Hence, it holds that

$$\begin{aligned}
\max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X) &= W_{\mathbb{P}^*}(\tilde{X}) \\
&\leqslant \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(\tilde{X}) \\
&\leqslant \max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X) \\
&\leqslant \max_{X\in\mathcal{Y}_{x_0}} W_{\mathbb{P}^*}(X) \\
&\leqslant \max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X),
\end{aligned}$$

in which the last inequality follows by $\mathcal{Y}_{x_0} \subset \mathcal{Y}_{x_0}^{\mathbb{P}^*}$. □

Assume that $\mathbb{F} = \mathbb{F}_{SSD}$ in Theorem 4.16. In this case, the requirement that $\tilde{X} \in \mathcal{Y}_{x_0}$ can be $\mathbb{P}^*-$a.s. be expressed as $f(\ell^*)$ for some $f \in \mathbb{F}_{SSD}$ is equivalent to the concavity of such f. This property is difficult to verify ex-ante. However, in the case of $W_{\mathbb{P}}$ being an expected utility, this condition translates into a condition on the utility function (see Theorem 4.18 below), which is easy to verify. To prove it, we need a technical lemma.

Lemma 4.17. *Let $(a,b) \subset \mathbb{R}$, where $a,b \in \mathbb{R} \cup \{\pm\infty\}$. Let $u : (a,b) \to \mathbb{R}$ be a differentiable, concave and strictly increasing function such that u' is strictly decreasing. It holds that $\frac{1}{u'}$ is convex if and only if $x \mapsto [u']^{-1}(\frac{1}{x})$ is concave. If u is three times differentiable, this condition is equivalent to*

$$a(x) \geqslant \frac{p(x)}{2}, \tag{4.8}$$

in which $a(x) := -\frac{u''(x)}{u'(x)}$ refers to the absolute risk aversion measure and $p(x) := -\frac{u'''(x)}{u''(x)}$ to the absolute prudence.

Proof. Note that $u' > 0$ and that $\frac{1}{u'}$ is strictly increasing. Observe that the inverse of $\frac{1}{u'}$ is $x \mapsto [u']^{-1}(\frac{1}{x})$, which is hence also strictly increasing. The inverse of a convex (concave) and strictly increasing function is concave (convex). For the second assertion, observe that $u'' < 0$ and that a function is convex on an open interval if and only if its second derivative is non-negative. □

Theorem 4.18. *Let $\mathbb{P} \in \mathcal{P}$ with likelihood ratio $\ell^{\mathbb{P}}$. Let $(a,b) \subset \overline{\mathbb{R}}$. Let $u : (a,b) \to \mathbb{R}$ be a differentiable, concave and strictly increasing utility function such that u' is strictly decreasing. Assume that for x_0 in Equation (4.3) it holds that $x_0 \in (a,b)$. If the maximization problem (4.2) under $\mathbb{P}$ has a solution, then the solution is a non-decreasing and concave function of $\ell^{\mathbb{P}}$ if and only if $\frac{1}{u'}$ is convex.*

Proof. By Lemma 2 in Bernard et al. (2015), the solution is unique and given by $[u']^{-1}\left(\frac{c_0}{\ell^{\mathbb{P}}}\right)$ for some $c_0 > 0$. See also Merton (1975) for a proof in a context in which Inada's conditions are satisfied. An application of Lemma 4.17 finishes the proof. □

Corollary 4.19. *Let $\mathbb{P} \in \mathcal{P}$ be a least favourable measure with respect to $\mathcal{F}_{SSD}$. Let $(a,b) \subset \overline{\mathbb{R}}$. Let $u : (a,b) \to \mathbb{R}$ be a differentiable, concave and strictly increasing utility function such that u' is strictly decreasing. Assume for x_0 in Equation (4.3) that it holds that $x_0 \in (a,b)$. If the maximization problem (4.2) has a solution under $\mathbb{P}$, then it also solves the robust maximization problem* (4.1).

Remark 4.20. Maggi et al. (2006) have shown that $a(x) > p(x)$ if and only if the utility has increasing absolute risk aversion, which is somewhat unusual (typical agents have decreasing absolute risk aversion given they become less risk averse as their wealth increases). Here, our condition is not incompatible with decreasing absolute risk aversion due to the factor $\frac{1}{2}$.

In addition, the condition in Equation (4.8) has appeared several times in the literature. It has been found to play a role in the context of insurance models with moral hazard in **?**Bourlès (2017), but it also appeared as a condition in the opening of a new asset market (Gollier and Kimball (1996)), when there is uncertainty on the size (Gollier et al. (2000)) or the probability of losses (Gollier (2002)) and under contingent auditing (Sinclair-Desgagné and Gabel (1997)). Further interpretation of this condition and, in particular, of the degree of concavity of the inverse of the marginal utility can be found in Bourlès (2017). This condition also appears in Varian (1985) in the context of portfolio selection under ambiguity.

Example 4.21. As an illustration, we provide two utility functions, which are differentiable, concave and strictly increasing functions such that one over the marginal utility is convex.

- The exponential utility for risk-averse agents: $u : (0, \infty) \to \mathbb{R}$, $x \mapsto 1 - e^{-\lambda x}$, for $\lambda > 0$. It holds that $\frac{1}{u'(x)} = e^{\lambda x}$, which is strictly increasing and convex.
- CRRA utility: $u : (0, \infty) \to \mathbb{R}$, $x \mapsto \frac{x^{1-\eta}}{1-\eta}$, for $\eta > 1$. It holds that $\frac{1}{u'(x)} = x^{\eta}$, which is strictly increasing and convex.

Example 4.22. A so-called *discount certificate* is a trading strategy, in which the agent purchases the underling for price S_0 today and sells a call option with strike $K > S_0$ at the same time. The agent acquires the underling at a discount, expressed by the premium π of the option, and, in exchange, accepts a cap on the potential return. Her maximal absolute return is $K - (S_0 - \pi)$. If the set $\mathbb{F}_{SSD}$ meets the regularity condition and the likelihood ratio is an increasing and concave function of S_T as in Corollary 3.16 and in the Examples (see Section 3.3.2), the discount certificate is non-decreasing and concave in the likelihood ratio.

5 Rationalizing robust cost-efficient payoffs

When there is no ambiguity on the probability measure $\mathbb{P}$, there is a close relation between cost-efficiency and portfolio optimization: for any cost-efficient payoff X, there exists a (generalized) utility function u (unique up to a linear transformation) such that X also solves the expected utility maximization problem (Bernard et al. (2015)). In this section, we show that this result can be generalized to the robust setting developed previously in that robust cost-efficient payoffs can be rationalized in terms of the maxmin utility framework introduced in Gilboa and Schmeidler (1989). Specifically, we show − under the same assumptions as in Theorems 4.12 and 4.16 − that payoffs maximize a robust utility functional as in Gilboa and Schmeidler (1989) if and only if they are robust cost-efficient.

5.1 In the absence of ambiguity

Consider the following *expected utility maximization problem under* $\mathbb{P}$:

$$\max_{X \in \mathcal{Y}^{\mathbb{P}}_{x_0,u}} E_{\mathbb{P}}(u(X)), \tag{5.1}$$

where $u \in \tilde{\mathcal{U}}_{\mathcal{I}}$ for some interval $\mathcal{I} \subset \overline{\mathbb{R}}$ and $\mathcal{Y}^{\mathbb{P}}_{x_0,u}$ is defined as in Definition 4.4.

Lemma 3 in Bernard et al. (2015) provides an explicit expression of the unique solution to the expected utility maximization problem (5.1). The following lemmas restates that result.

Lemma 5.1. *Assume that $\ell^{\mathbb{P}}$ is continuously distributed under $\mathbb{P}$. Let $\mathcal{I} = (a, b) \subset \overline{\mathbb{R}}$ be an open interval and let $u \in \tilde{\mathcal{U}}_{\mathcal{I}}$ such that $x_0 := E_{\mathbb{Q}}\left[[u']^{-1}\left(1/\ell^{\mathbb{P}}\right)\right] < +\infty$ and where $[u']^{-1}$ is a generalized inverse on $\mathcal{I}$, $[u']^{-1}(y) = \inf\{x \in \mathcal{I} : [u'](x) \leq y\}$ with the convention that $\inf\{\emptyset\} = \sup\{\mathcal{I}\}$. The optimal solution to* (5.1) *exists, is $\mathbb{P}$-a.s. unique and is given by*

$$X^* = [u']^{-1}\left(\frac{1}{\ell^{\mathbb{P}}}\right), \ \ \mathbb{P} - a.s.$$

If X is cost-efficient, then X also solves some expected utility maximization problems, as the next lemma shows.

Lemma 5.2. *Assume that $\ell^{\mathbb{P}}$ is continuously distributed under $\mathbb{P}$. Let X^* be a $\mathbb{P}-$cost-efficient payoff with distribution $F^{\mathbb{P}}_{X^*}$. Define $\mathcal{I} = \{x : F^{\mathbb{P}}_{X^*}(x) \in (0,1)\}$: then it is an interval $\mathcal{I} \subset \overline{\mathbb{R}}$. Assume that $a := \inf\{\mathcal{I}\} > -\infty$ and let $c \in \mathcal{I}$. The utility function defined by*

$$u(x) = \begin{cases} \int_c^x F^{-1}_{\frac{1}{\ell^{\mathbb{P}}}}(1 - F^{\mathbb{P}}_{X^*}(y))dy & \text{if } x \in [a, \infty) \\ -\infty & \text{if } x < a, \end{cases} \tag{5.2}$$

where we use the conventions $F^{-1}_{\frac{1}{\ell^{\mathbb{P}}}}(0) = 0$ and $F^{-1}_{\frac{1}{\ell^{\mathbb{P}}}}(1) = \infty$ belongs to the set of generalized utility functions $\tilde{\mathcal{U}}_{\mathcal{I}}$. Then, X^* is the solution to the expected utility maximization problem (5.1) *under $\mathbb{P}$ for this utility function u.*

Proof. The claim follows as in the proof of Theorem 3 in Bernard et al. (2015). □

5.2 In the presence of ambiguity

In the following two theorems the sets $\mathcal{Y}_{x_0}$, $\mathcal{Y}^{\mathbb{P}^*}_{x_0}$, $\mathcal{Y}_{x_0,u}$ and $\mathcal{Y}^{\mathbb{P}^*}_{x_0,u}$ are defined as in Definition 4.4.

Theorem 5.3. *Assume that the set $\mathbb{F}_{FSD}$ meets the regularity condition. Let X^* be a payoff bounded from below. Let $\mathcal{I} \subset \overline{\mathbb{R}}$ be a non-empty interval describing the support of $F_{X^*}^{\mathbb{P}^*}$. Let $x_0 = E_{\mathbb{Q}}[X^*]$ and assume that x_0 lies in the interior of $\mathcal{I}$. Let $a = \inf(\mathcal{I})$ and $c > 0$, such that $F_{X^*}^{\mathbb{P}^*}(c) > 0$, and define*

$$u(x) = \begin{cases} \int_c^x F_{\frac{1}{\ell^*}}^{-1}\left(1 - F_{X^*}^{\mathbb{P}^*}(y)\right) dy & , x \in [a, \infty) \\ -\infty & , x < a. \end{cases}$$

Assume that $E_{\mathbb{P}}[u(X^*)] < \infty$ for all $\mathbb{P} \in \mathcal{P}$. The following statements are equivalent:

i) *X^* is cost-efficient under $\mathbb{P}^*$.*

ii) *It holds that $X^* = \left[F_{X^*}^{\mathbb{P}^*}\right]^{-1}\left(F_{\ell^{\mathbb{P}^*}}^{\mathbb{P}^*}\left(\ell^{\mathbb{P}^*}\right)\right)$, $\mathbb{P}^*-a.s.$*

iii) *X^* is $\mathbb{P}^*-a.s.$ non-decreasing in ℓ^*.*

iv) *X^* solves the robust FSD-cost-efficiency problem for $F_{X^*}^{\mathbb{P}^*}$.*

v) *X^* solves the expected utility maximization problem under $\mathbb{P}^*$ for the generalized utility function u*

$$\max_{X \in \mathcal{Y}_{x_0,u}^{\mathbb{P}^*}} E_{\mathbb{P}^*}(u(X)).$$

vi) *X^* is the unique solution to the robust expected utility problem for the generalized utility function u*

$$\max_{X \in \mathcal{Y}_{x_0,u}} \inf_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}(u(X)).$$

vii) *X^* is the unique solution to the maxmin expected utility problem for the generalized utility function u*

$$\max_{X \in \mathcal{Y}_{x_0,u}} \min_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}(u(X)).$$

viii) *There is a law invariant family of preference $(W_{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}}$, which is FSD-family consistent on $\mathcal{Y}_{x_0}$ with respect to $\mathbb{P}^*$ such that X^* is the unique solution to the maximization problem under $\mathbb{P}^*$*

$$\max_{X \in \mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X)$$

and also $X^* \in \mathcal{Y}_{x_0}$.

ix) *There is a law invariant family of preference $(W_{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}}$, which is FSD-family consistent on $\mathcal{Y}_{x_0}$ with respect to $\mathbb{P}^*$ such that X^* is the unique solution to the robust maximization problem*

$$\max_{X \in \mathcal{Y}_{x_0}} \inf_{\mathbb{P} \in \mathcal{P}} W_{\mathbb{P}}(X).$$

Proof. By Propositions 3.8 and 3.11, $\mathbb{F}_{FSD}$ is composition-consistent and is cost-consistent. The equivalence between i), ii) and iii) follows from Lemma 2.8. The equivalence between iv) and ii) follows from Theorem 3.17. By Lemma 5.2, i) implies v). By Corollary 4.14 and Lemma 5.1, v) implies i). By Lemma 4.10, v) implies viii) trivially, define $W_{\mathbb{P}}(.) = E(u(.))$ for all $\mathbb{P} \in \mathcal{P}$. v) implies vi) by Theorem 4.12 and Lemma 4.10 as $E_{\mathbb{P}}[u(X^*)] < \infty$ for all $\mathbb{P} \in \mathcal{P}$ by assumption. vi) implies ix) trivially. vi) and vii) are equivalent because the infimum in vi) is attained at $\mathbb{P}^*$. By Corollaries 4.14 and 4.15, vi)⇒i) and viii)⇒i) and ix)⇒i). □

Theorem 4.16 generalizes Theorem 4.12 to general preferences - which are not law-invariant - at the cost of assuming a priori that solutions have a certain structure. Similarly, the following Theorem 5.4 generalizes Theorem 5.3 by dropping the assumption of law-invariance and of the uniqueness of the solution, requiring instead that the solution can be expressed as $f(\ell^*)$ for some $f \in \mathbb{F}$.

Theorem 5.4. *Let $\mathbb{F}$ be a set of measurable functions such that $\mathbb{F}_{SSD} \subset \mathbb{F} \subset \mathbb{F}_{FSD}$ and $\mathbb{F}$ meets the regularity condition. Let X^* and u be as in Theorem 5.3. We further assume $\left[F_{X^*}^{\mathbb{P}^*}\right]^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}$. The following statements are equivalent:*

i) *X^* is cost-efficient under $\mathbb{P}^*$.*

ii) *It holds that $X^* = \left[F_{X^*}^{\mathbb{P}^*}\right]^{-1}\left(F_{\ell^{\mathbb{P}^*}}^{\mathbb{P}^*}\left(\ell^{\mathbb{P}^*}\right)\right)$, $\mathbb{P}^*-a.s.$*

iii) *X^* is $\mathbb{P}^*-a.s.$ non-decreasing in ℓ^*.*

iv) *X^* solves the robust $\mathbb{F}$-cost-efficiency problem for $F_{X^*}^{\mathbb{P}^*}$.*

v) *X^* solves the expected utility maximization problem under $\mathbb{P}^*$ for the generalized utility function u*

$$\max_{X \in \mathcal{Y}_{x_0,u}^{\mathbb{P}^*}} E_{\mathbb{P}^*}(u(X))$$

and $X^* = f(\ell^*)$, $\mathbb{P}^*-a.s.$ for some $f \in \mathbb{F}$.

vi) *X^* solves the robust expected utility problem for the generalized utility function u*

$$\max_{X \in \mathcal{Y}_{x_0,u}} \inf_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}(u(X))$$

and $X^* = f(\ell^*)$, $\mathbb{P}^*-a.s.$ for some $f \in \mathbb{F}$.

vii) *X^* solves the maxmin expected utility problem for the generalized utility function u*

$$\max_{X \in \mathcal{Y}_{x_0,u}} \min_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}(u(X))$$

and $X^* = f(\ell^*)$, $\mathbb{P}^*-a.s.$ for some $f \in \mathbb{F}$.

viii) *There is a family of preference* $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$*, which is* $\mathbb{F}$*-family consistent on* $\mathcal{Y}_{x_0}$ *with respect to* $\mathbb{P}^*$ *such that* X^* *is the solution to the maximization problem under* $\mathbb{P}^*$

$$\max_{X\in\mathcal{Y}_{x_0}^{\mathbb{P}^*}} W_{\mathbb{P}^*}(X)$$

and $X^* = f(\ell^*)$*,* $\mathbb{P}^*-a.s.$ *for some* $f \in \mathbb{F}$ *and* $X^* \in \mathcal{Y}_{x_0}$*.*

ix) *There is a family of preference* $(W_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$*, which is* $\mathbb{F}$*-family consistent on* $\mathcal{Y}_{x_0}$ *with respect to* $\mathbb{P}^*$ *such that* X^* *is the solution to the robust maximization problem*

$$\max_{X\in\mathcal{Y}_{x_0}} \inf_{\mathbb{P}\in\mathcal{P}} W_{\mathbb{P}}(X)$$

and $X^* = f(\ell^*)$*,* $\mathbb{P}^*-a.s.$ *for some* $f \in \mathbb{F}$*.*

Proof. By Proposition 3.11, $\mathbb{F}$ is cost-consistent. The implications i)⇔ii)⇔iii)⇔iv) and v)⇒viii) and vi)⇒ix) and vi)⇔vii) follow as in the proof of Theorem 5.3. By Lemma 5.2, i) and ii) imply v). By Theorem 4.16, v) implies vi). As $f \in \mathbb{F} \subset \mathbb{F}_{FSD}$ is non-decreasing, v) through ix) imply iii) trivially. □

Remark 5.5. Recall that $\left[F_{X^*}^{\mathbb{P}^*}\right]^{-1} \circ F_{\ell^*}^{\mathbb{P}^*} \in \mathbb{F}_{FSD}$ is always true.

Remark 5.6. It makes sense that all functions in $\mathbb{F}$ are non-decreasing, i.e., that $\mathbb{F} \subset \mathbb{F}_{FSD}$. Otherwise, there are two sure payoffs $x_0, y_0 \in \mathcal{X}$, i.e., x_0, y_0 are constant, such that $x_0 < y_0$ but the distribution of y_0 does not dominate the distribution of x_0 in integral stochastic ordering.

Remark 5.7. The preferences in Theorem 5.4 do not need to be law-invariant or increasing, i.e., $X \leqslant Y$ does not need to imply $W_{\mathbb{P}}(X) \leqslant W_{\mathbb{P}}(Y)$. We provide a simple example of such a preference. Define $X^* = f(\ell^*)$ for some $f \in \mathbb{F}$. Let $x_0 = E_{\mathbb{Q}}[X^*]$. For $\mathbb{P} \in \mathcal{P}$, define

$$W_{\mathbb{P}}(X) = \begin{cases} 1 & , X = X^*, \ \mathbb{P}-\text{a.s.} \\ 0 & , \text{ otherwise,} \end{cases}$$

which is trivially $\mathbb{F}-$family consistent. An agent with such a preference only likes X^* and neglects everything else. She is not law-invariant and does not prefer more to less. Someone interested only in the market portfolio or in the risk-free bond might have such a preference. The solution to the robust maximization problem ix) is X^*, which is cost-efficient because X^* is non-decreasing in ℓ^*. A generalized utility function for v) can be constructed as in Equation 5.2.

Example 5.8. We provide a trivial example, which gives some insight nevertheless. Let $W_{\mathbb{P}}(.) = 0$ for all $\mathbb{P} \in \mathcal{P}$. Then, any payoff $X \in \mathcal{X}$ solves the optimization problem in viii) or ix). Take a solution $X^* := f(\ell^*)$ for some $f \in \mathbb{F}$ where f is non-decreasing. That solution is cost-efficient, and a generalized utility function for v) and vi) can be constructed as in Equation 5.2.

6 Final Remarks

In this paper we assume the agent has Knightian uncertainty. He is unsure about the precise physical measure describing the financial market and knows only that the true physical measure lies within a set $\mathcal{P}$ of probability measures. Given this ambiguity, it is no longer possible to target a payoff with a given probability distribution function. In particular, the close relation between payoffs that are the cheapest possible in reaching a target distribution and the optimality thereof under law-invariant increasing preferences (**?**Dybvig (1988), Sharpe et al. (2000), Goldstein et al. (2008), Bernard et al. (2015)) is a priori lost as there is no consensus regarding what probability distribution to adopt. For this reason, we introduce the notion of *robust cost-efficient payoff*.

For a given distribution function F_0 the robust cost-efficiency problem aims at finding the cheapest payoff whose distribution dominates F_0 under all possible physical measures in some integral stochastic ordering. We solve this problem under some conditions (namely, where there exists a least favourable measure $\mathbb{P}^*$ and the integral stochastic ordering $\preceq_{\mathbb{F}}$ is cost-consistent). The solution is identical to the solution to the cost-efficiency problem without model ambiguity under the physical measure $\mathbb{P}^*$ and given in closed-form. We are thus able to reduce the problem formulated in a robust setting to a problem formulated in a standard setting without model ambiguity.

Finally, we show that this notion of robust cost-efficiency plays a key role in optimal robust portfolio selection and that a very general class of robust portfolio selection problems (possibly in a non-expected utility setting) can be reduced to the maxmin expected utility setting of Gilboa and Schmeidler (1989) for a well-chosen concave utility function. For this to hold, we make a relatively minor assumption on the family of preferences, i.e., that it is family consistent: if the measure $\mathbb{P}^*$ is the most pessimistic view of a payoff Y, then the preference under that measure is the lowest as well. To the best of our knowledge, family consistency is new to the literature, and we provide several examples in the context of expected utility theory and Yaari's Dual Theory.

A Proof of Proposition 2.2

Proof. Abbreviate S_T by S. Let $\varepsilon > 0$. Let $u > 0$ such that $E_{\mathbb{Q}}\left[|g(S) - g_u(S)|\right] < \frac{\varepsilon}{3}$, where $g_u(x) = 1_{\{x \leqslant u\}} g(x)$. The function g_u has compact support and is square integrable by assumption: hence, there is a step function

$$\psi(x) = \sum_{i=1}^{M} \alpha_i 1_{[0,K_i]}(x), \ \ \alpha_i \in \mathbb{R}, \ \ K_i \geqslant 0, \ \ M \in \mathbb{N}$$

such that

$$\int_0^\infty |\psi(s) - g_u(s)|^2 \, ds < \frac{\varepsilon}{3\Lambda},$$

where $\Lambda = \int_0^\infty \left| f_S^{\mathbb{Q}}(s) \right|^2 ds$. The number Λ is finite because $f_S^{\mathbb{Q}}$ is a bounded density: thus, it is square-integrable. It follows by Cauchy-Schwarz inequality that

$$E_{\mathbb{Q}}\left[|\psi(S) - g_u(S)|\right] = \int_0^\infty |\psi(s) - g_u(s)| \, f_S^{\mathbb{Q}}(s) ds < \frac{\varepsilon}{3}.$$

The payoff-function $1_{[0,K_i]}(x)$ corresponds to the payoff function of a digital option with strike K_i. It can be approximated arbitrarily closely by trading a bull call spread: go $N_i \in \mathbb{N}$ call options long with strike K_i and the same number of call options short with the higher strike $K_i + \frac{1}{N_i}$. A bull call spread corresponds to the payoff function

$$h_i(x) = N_i \left[(x - K_i)^+ - \left(x - \left(K_i + \frac{1}{N_i} \right) \right)^+ \right].$$

It follows that

$$\left| 1_{[0,K_i]}(x) - h_i(x) \right| \leqslant \begin{cases} 1 & , x \in \left[K_i, K_i + \frac{1}{N_i} \right) \\ 0 & , \text{ otherwise,} \end{cases}$$

and therefore we can choose N_i large enough that

$$E_{\mathbb{Q}}\left[|\alpha_i| \left| 1_{[0,K_i]}(S) - h_i(S) \right| \right] \leqslant |\alpha_i| \, \mathbb{Q}\left(S \in \left[K_i, K_i + \frac{1}{N_i} \right) \right) \leqslant \frac{\varepsilon}{3M}.$$

In conclusion, there are $N_1, ..., N_M \in \mathbb{N}$ such that

$$E_{\mathbb{Q}}\left[\left| \psi(S) - \sum_{i=1}^M \alpha_i h_i(S) \right| \right] < \frac{\varepsilon}{3}.$$

It follows that

$$\begin{aligned} E_{\mathbb{Q}}\left[\left| \sum_{i=1}^M \alpha_i h_i(S) - g(S) \right| \right] &\leqslant E_{\mathbb{Q}}\left[\left| \sum_{i=1}^M \alpha_i h_i(S) - \psi(S) \right| \right] \\ &\quad + E_{\mathbb{Q}}\left[|\psi(S) - g_u(S)| \right] \\ &\quad + E_{\mathbb{Q}}\left[|g(S) - g_u(S)| \right] \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, there exists a static portfolio of call options converging to $g(S)$ in mean with respect to $\mathbb{Q}$. Assume $\mathbb{P}(g(S) \geqslant 0) \geqslant 0$ and $\mathbb{P}(g(S) > 0) > 0$ for some $\mathbb{P} \in \mathcal{P}$. The same holds for $\mathbb{Q}$: therefore, $E_{\mathbb{Q}}[g(S)] > 0$. The approximating portfolio cannot be an arbitrage because its price is greater than zero for large M, as

$$\lim_{M \to \infty} E_{\mathbb{Q}}\left[\sum_{i=1}^M \alpha_i h_i(S) \right] = E_{\mathbb{Q}}[g(S)] > 0.$$

Therefore, the price of $g(S)$ must be $E_{\mathbb{Q}}[g(S)]$: otherwise, there would be an arbitrage opportunity. Hence $g(S_T) \in \mathcal{X}_e$. $\square$

B Direct Proof of Cost-Consistency of $\mathbb{F}_{FSD}$

As discussed immediately prior to Proposition 3.11, this proof is not needed in the case of FSD. The following lemma is useful to prove that $\mathbb{F}_{FSD}$ is cost-consistent.

Lemma B.1. *Let $X, Y \in \mathcal{X}$. If $X \leqslant Y$, $\mathbb{Q}-$a.s. and $\mathbb{Q}(X \neq Y) > 0$, then $E_{\mathbb{Q}}[X] < E_{\mathbb{Q}}[Y]$.*

Proof. Let $Z = Y - X$. Then $Z \geqslant 0$, $\mathbb{Q}-$a.s. For $n \in \mathbb{N}$, let $A_n = \{Z \geqslant \frac{1}{n}\}$. Then $Z \geqslant \frac{1}{n} 1_{A_n}$ and $E_{\mathbb{Q}}[Z] \geqslant \frac{1}{n}\mathbb{Q}(A_n)$. If $E_{\mathbb{Q}}[Z] = 0$, we would have $\mathbb{Q}(A_n) = 0$ for all $n \in \mathbb{N}$, and thus $\{Z \neq 0\} = \cup_{n \in \mathbb{P}} A_n$ would have measure zero, which contradicts $\mathbb{Q}(X \neq Y) > 0$. □

Proposition B.2. *$\mathbb{F}_{FSD}$ is $\mathbb{P}-$cost-consistent for all $\mathbb{P} \in \mathcal{P}$.*

Proof. Let $\mathbb{P} \in \mathcal{P}$ and $X, Y \in \mathcal{X}$. *Step 1:* Assume $F_X^{\mathbb{P}} \preceq_{\mathbb{F}_{FSD}} F_Y^{\mathbb{P}}$. Then $\left[F_Y^{\mathbb{P}}\right]^{-1} \leqslant \left[F_X^{\mathbb{P}}\right]^{-1}$. Because X and Y are $\mathbb{P}-$cost-efficient, it holds that

$$X = \left[F_X^{\mathbb{P}}\right]^{-1}\left(F_{\ell^{\mathbb{P}}}^{\mathbb{P}}(\ell^{\mathbb{P}})\right) \leqslant \left[F_Y^{\mathbb{P}}\right]^{-1}\left(F_{\ell^{\mathbb{P}}}^{\mathbb{P}}(\ell^{\mathbb{P}})\right) = Y, \;\; \mathbb{P} - \text{a.s.}$$

Hence, $X \leqslant Y$, $\mathbb{Q}-$a.s. because $\mathbb{P}$, and $\mathbb{Q}$ are equivalent and therefore $E_{\mathbb{Q}}[X] \leqslant E_{\mathbb{Q}}[Y]$. *Step 2:* Assume additionally that $F_X^{\mathbb{P}}(x) \neq F_Y^{\mathbb{P}}(x)$ for some $x \in \mathbb{R}$. Let $A = \{X \neq Y\}$. It then must hold that $\mathbb{P}(A) > 0$. As $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, it follows that $\mathbb{Q}(A) > 0$, and thus $E_{\mathbb{Q}}[X] < E_{\mathbb{Q}}[Y]$ by Lemma B.1. □

C Proof of Propositions 3.13 and 3.14

Proof of Proposition 3.13. We begin by showing that ℓ^* being the convex minimal element in $\mathcal{L}$ implies that $\mathbb{P}^*$ is a least favourable measure. From Theorem 2.1 in Denuit and Müller (2002) and the subsequent illustrations in Section 3 therein it follows immediately that the inequality

$$\mathbb{E}_{\mathbb{Q}}[v(\ell^*)] = \int v(x) dF_{\ell^*}^{\mathbb{Q}} \leqslant \int v(x) dF_{\ell}^{\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}}[v(\ell)]$$

for some $\ell \in \mathcal{L}$ holds for all convex functions $v \colon \mathbb{R} \to \mathbb{R}$ if and only if it holds for the subset of all convex, bounded and infinitely differentiable functions $v \in \mathcal{C}_b^\infty$. Hence, our assumption implies, due to the first-order optimality condition, that for any such v we have

$$\mathbb{E}_{\mathbb{Q}}[v'(\ell^*)(\ell - \ell^*)] \geqslant 0 \quad \text{for all } \ell \in \mathcal{L}. \tag{C.1}$$

Moreover, for any non-decreasing function f in $\mathcal{C}_b^\infty$ there exists a convex $v_f \in \mathcal{C}_b^\infty$ such that $v_f' = f$. Hence, for every non-decreasing f in $\mathcal{C}_b^\infty$ it holds that

$$\mathbb{E}_{\mathbb{P}}[f(\ell^*)] - \mathbb{E}_{\mathbb{P}^*}[f(\ell^*)] = \mathbb{E}_{\mathbb{Q}}[f(\ell^*)(\ell - \ell^*)] = \mathbb{E}_{\mathbb{Q}}[v_f'(\ell^*)(\ell - \ell^*)] \geqslant 0$$

where the last inequality follows from C.1. Thus,

$$\mathbb{E}_{\mathbb{P}}[f(\ell^*)] = \int f(x)dF^{\mathbb{P}}_{\ell^*} \geqslant \int f(x)dF^{\mathbb{P}^*}_{\ell^*} = \mathbb{E}_{\mathbb{P}^*}[f(\ell^*)]$$

for all non-decreasing $f \in \mathcal{C}_b^\infty$ and using again Theorem 2.1 in Denuit and Müller (2002), the inequality in fact holds for all non-decreasing functions f such that the expectations exist, showing that $\mathbb{P}^*$ is indeed a least favourable measure.

For the converse direction, assume that $\mathbb{P}^*$ is a least favourable measure: hence,

$$\mathbb{E}_{\mathbb{P}}[f(\ell^*)] = \int f(x)dF^{\mathbb{P}}_{\ell^*} \geqslant \int f(x)dF^{\mathbb{P}^*}_{\ell^*} = \mathbb{E}_{\mathbb{P}^*}[f(\ell^*)]$$

for all non-decreasing $f \in \mathcal{C}_b^\infty$. Using the fact that every convex v_f has a non-decreasing derivative $v_f' = f$ we obtain

$$0 \leqslant \mathbb{E}_{\mathbb{P}}[f(\ell^*)] - \mathbb{E}_{\mathbb{P}^*}[f(\ell^*)] = \mathbb{E}_{\mathbb{Q}}[f(\ell^*)(\ell - \ell^*)] = \mathbb{E}_{\mathbb{Q}}[v_f'(\ell^*)(\ell - \ell^*)]$$

and the claims follow again by the first-order optimality condition. □

Proof of Proposition 3.14. To show the equivalence in the SSD-case we need to establish the equivalence between

$$\mathbb{E}_{\mathbb{P}}[f(\ell^*)] = \int f(x)dF^{\mathbb{P}}_{\ell^*} \geqslant \int f(x)dF^{\mathbb{P}^*}_{\ell^*} = \mathbb{E}_{\mathbb{P}}[f(\ell^*)] \quad \text{for all } \mathbb{P} \in \mathcal{P}$$

for all non-decreasing and concave functions $f\colon \mathbb{R} \to \mathbb{R}$ and

$$\mathbb{E}_{\mathbb{Q}}[v(\ell^*)] = \int v(x)dF^{\mathbb{Q}}_{\ell^*} \leqslant \int v(x)dF^{\mathbb{Q}}_{\ell} = \mathbb{E}_{\mathbb{Q}}[v(\ell)] \quad \text{for all } \ell \in \left\{\frac{d\mathbb{P}}{d\mathbb{Q}} : \mathbb{P} \in \mathcal{P}\right\}$$

for all convex and differentiable $v\colon \mathbb{R} \to \mathbb{R}$ with concave derivative v'. Note that for every concave and non-decreasing $f \in \mathcal{C}_b^\infty$ there exists a convex $v \in \mathcal{C}_b^\infty$ with $v' = f$. The proof then follows along the same lines as the proof of Proposition 3.13. □

D Proof of Proposition 3.20

Lemma D.1. *Let F and G be two cdf. It holds that $G \preceq_{\mathbb{F}_{TSD}} F$ if and only if*

$$\int_{-\infty}^{\eta}\int_{-\infty}^{\xi} F(x)dxd\xi \leqslant \int_{-\infty}^{\eta}\int_{-\infty}^{\xi} G(x)dxd\xi, \ \ \eta \in \mathbb{R}.$$

Proof. See Theorem 2.2 of Gotoh and Konno (2000). □

We are now ready to prove Proposition 3.20.

Proof. Apply the chain rule to show that $\mathbb{F}_{TSD}$ is composition-consistent. Next, we construct two distributions such that one dominates the other in TSD but is cheaper.

Step 1: define some market setting as in Section 3.3.1: let $\mu_1 = 0.01$, $r = 0$, $T = 1$ and $s = 0.1$. Then $\frac{\mu_1 - r}{s^2} = 1$. Choose S_0 such that it holds that $\ell^* := \ell^{\mu_1,s} = S_T$: see Equation (3.4), i.e., $\log(S_0) = -0.0025$. Under $\mathbb{P}^* := \mathbb{P}^{\mu_1,s}$ the stock is log-normal distributed with parameters $(\mu_1 - \frac{s^2}{2} + \log(S_0)) = 0.0025$ and $s^2 = 0.01$. Under $\mathbb{Q}$, the stock is also log-normal distributed with parameters $(r - \frac{s^2}{2} + \log(S_0)) = -0.0075$ and $s^2 = 0.01$. $\mathbb{P}^*$ is a least favourable measure with respect to the set $\mathbb{F}_{FSD}$, and hence also $\mathbb{F}_{TSD}$ because $\mathbb{F}_{TSD} \subset \mathbb{F}_{FSD}$.

Step 2: define two distributions: Let

$$F(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \leqslant x < 1 \\ 1 & , x \geqslant 1 \end{cases}$$

and, for $p_0 \in (0, 1)$, let

$$G(x) = \begin{cases} 0 & , x < 0 \\ p_0 & , 0 \leqslant x < 1 \\ 1 & , x \geqslant 1. \end{cases}$$

F is the uniform distribution and G jumps at zero and at one. It follows that $F^{-1}(p) = p$ and that

$$G^{-1}(p) = \begin{cases} 0 & , p \in (0, p_0] \\ 1 & , p > p_0. \end{cases}$$

Step 3: show that F dominates G in TSD: It holds for $\eta \in (0, 1)$ that

$$\int_{-\infty}^{\eta} \int_{-\infty}^{\xi} F(x) dx d\xi = \int_{0}^{\eta} \int_{0}^{\xi} x dx d\xi = \frac{1}{6}\eta^3.$$

and that

$$\int_{-\infty}^{\eta} \int_{-\infty}^{\xi} G(x) dx d\xi = \int_{0}^{\eta} \int_{0}^{\xi} p_0 dx d\xi = \frac{1}{2} p_0 \eta^2.$$

Hence, if $\frac{1}{6}\eta^3 \leqslant \frac{1}{2}p_0\eta^2$ or, equivalently $p_0 \geqslant \frac{1}{3}$, it follows that $G \preceq_{TSD} F$.

Step 4: compute the lowest cost of both distributions: The cost-efficient payoff for F is

$$X_F = F^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right) = F_{S_T}^{\mathbb{P}^*}(S_T).$$

The lowest price of F can be computed numerically:

$$E_{\mathbb{Q}}[X_F] = \int_0^{\infty} F_{S_T}^{\mathbb{P}^*}(s) f_{S_T}^{r}(s) ds = 0.472.$$

The cost-efficient payoff for G is

$$X_G = G^{-1}\left(F_{\ell^*}^{\mathbb{P}^*}(\ell^*)\right) = \begin{cases} 1 & , S_T > \left[F_{S_T}^{\mathbb{P}^*}\right]^{-1}(p_0) \\ 0 & , \text{otherwise} \end{cases}.$$

Its price is

$$E_{\mathbb{Q}}[X_G] = \int_{\left[F_{S_T}^{\mathbb{P}^*}\right]^{-1}(p_0)}^{\infty} f_{S_T}^r(s)ds = 1 - F_{S_T}^{\mathbb{Q}}\left(\left[F_{S_T}^{\mathbb{P}^*}\right]^{-1}(p_0)\right).$$

Under $\mathbb{P}^*$, X_F is uniform distributed and X_G is a digital option. If $p_0 = \frac{1}{3}$, the lowest price for G is 0.63, which is greater than the lowest price to be paid for F. But in this case $G \preceq_{TSD} F$: hence, TSD is not cost-consistent. □

E Proof of Lemma 4.10

We use the convention $\pm\infty \cdot 0 = 0$.

Lemma E.1. *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\mathbb{P}^*$ and $\mathbb{P}$ be two equivalent probability measures. Let $X : \Omega \to \mathbb{R}$ be bounded from below and measurable. Let $f : \mathbb{R} \to \{-\infty\} \cup \mathbb{R}$ be non-decreasing. Then $E_{\mathbb{P}^*}[f(X)] = -\infty$ if and only if $E_{\mathbb{P}}[f(X)] = -\infty$.*

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of elementary functions converging to X from below: $v_n(\omega) = \sum_{i=1}^n \alpha_i 1_{A_i}(\omega)$ with $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{F}$ disjoint, $v_n(\omega) \leqslant v_{n+1}(\omega)$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} v_n(\omega) = X(\omega)$. Then it holds for all $n \in \mathbb{N}$ that

$$-\infty = E_{\mathbb{P}^*}[f(X)] \geqslant \sum_{i=1}^n f(\alpha_i)\mathbb{P}^*(A_i).$$

Hence, for all $n \in \mathbb{N}$ there must be a $j \leqslant n$ such that $f(\alpha_j) = -\infty$ and $\mathbb{P}^*(A_j) > 0$. This implies, by equivalence of $\mathbb{P}^*$ and $\mathbb{P}$, that

$$-\infty \geqslant \sum_{i=1}^n f(\alpha_i)\mathbb{P}(A_i) \quad \forall n \in \mathbb{N}.$$

Taking the supremum, it follows that $E_{\mathbb{P}}[f(X)] = -\infty$. The other direction follows similarly. □

The proof of Lemma 4.10 follows.

Proof. By Lemma E.1 both expectations $E_{\mathbb{P}}[u(Y)]$ and $E_{\mathbb{P}^*}[u(Y)]$ for $Y \in \mathcal{Y}_{x_0,u}$ are either equal to minus infinity or are both finite. Thus, without loss of generality, we assume that the expectations are finite for the rest of the proof. By Lemma 4.8 it is sufficient to show SSD-family consistency. Let $a := \inf(\mathcal{I})$.

Let $Y \in \mathcal{Y}_{x_0,u}$ such that $F_Y^{\mathbb{P}^*} \preceq_{SSD} F_Y^{\mathbb{P}}$ for some $\mathbb{P}^* \in \mathcal{P}$ and for all $\mathbb{P} \in \mathcal{P}$. We distinguish three cases. *Case 1:* $a = -\infty$. We have $E_{\mathbb{P}}[u(Y)] < \infty$ for all $\mathbb{P} \in \mathcal{P}$. Therefore, it holds that by Lemma 3.5

$$W_{\mathbb{P}^*}(Y) = E_{\mathbb{P}^*}[u(Y)] \leqslant E_{\mathbb{P}}[u(Y)] = W_{\mathbb{P}}(Y)$$

because u is non-decreasing and concave and $E_{\mathbb{P}}[u(X)]$ is finite for all $\mathbb{P} \in \mathcal{P}$. *Case 2:* $-\infty < a$ and $u(a) > -\infty$. Then $u \notin \mathbb{F}_{SSD}$. Thus, technically, the inequality (3.1) in Definition 3.1 cannot be applied. However, Definition 3.1 and Lemma 3.4 can be stated as well for functions mapping a possible finite interval to the reals: see Hadar and Russell (1969) or Meyer (1977) and, more generally, Wong and Li (1999).

Case 3: $-\infty < a$ and $u(a) = -\infty$. It follows that $\mathbb{P}(Y > a) = 1$ for all $\mathbb{P} \in \mathcal{P}$ because the expectations are finite. Let $\varepsilon > 0$. The function u is concave on (a, b) and admits a left-hand derivative u'_- at $a + \varepsilon$. Define

$$\tilde{u}_\varepsilon(x) = \begin{cases} u(x) & , x \geqslant \varepsilon + a \\ u'_-(a+\varepsilon)(x - (a+\varepsilon)) + u(a+\varepsilon) & , x < \varepsilon + a \end{cases}$$

then $\tilde{u}_\varepsilon \in \mathbb{F}_{SSD}$ and $E_{\mathbb{P}}[\tilde{u}_\varepsilon(Y)] \to E_{\mathbb{P}}[u(Y)]$ as $\varepsilon \to 0$ by monotone convergence.
□

References

Yacine Aït-Sahalia and Andrew W Lo. Nonparametric estimation of state-price densities implicit in financial asset prices. *The journal of finance*, 53(2):499–547, 1998.

Ole E Barndorff-Nielsen and Neil Shephard. Non-gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(2):167–241, 2001.

David S Bates. Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Review of Financial Studies*, 9(1):69–107, 1996.

Carole Bernard, Phelim Boyle, and Steven Vanduffel. Explicit representation of cost-efficient strategies. *Finance*, 35:5–55, 2014.

Carole Bernard, Jit Seng Chen, and Steven Vanduffel. Rationalizing investors' choices. *Journal of Mathematical Economics*, 59:10–23, 2015.

Carole Bernard, Steven Vanduffel, and Jiang Ye. Optimal strategies under Omega ratio. *European Journal of Operational Research*, 275(2):755–767, 2019.

Renaud Bourlès. Prevention incentives in long-term insurance contracts. *Journal of Economics & Management Strategy*, 26(3):661–674, 2017.

Douglas T. Breeden and Robert H. Litzenberger. Prices of state-contingent claims implicit in option prices. *Journal of Business*, 1(51):621–651, 1978.

Peter Carr and Dilip Madan. Towards a theory of volatility trading. In Robert A Jarrow, editor, *Volatility: New estimation techniques for pricing derivatives*, chapter 29, pages 417–427. Risk books, 1998.

Bent J Christensen and Nagpurnanand R Prabhala. The relation between implied and realized volatility. *Journal of financial economics*, 50(2):125–150, 1998.

Bent Jesper Christensen and Charlotte Strunk Hansen. New evidence on the implied-realized volatility relation. *The European Journal of Finance*, 8(2):187–205, 2002.

José Manuel Corcuera, Florence Guillaume, Peter Leoni, and Wim Schoutens. Implied Lévy volatility. *Quantitative Finance*, 9(4):383–393, 2009.

John C Cox and Hayne E Leland. On dynamic investment strategies. *Journal of Economic Dynamics and Control*, 24(11):1859–1880, 2000.

Michel Denuit and Alfred Müller. Smooth generators of integral stochastic orders. *The Annals of Applied Probability*, 12(4):1174–1184, 2002.

Michel Denuit, Jan Dhaene, Marc Goovaerts, and Rob Kaas. *Actuarial Theory for Dependent Risks*. John Wiley & Sons, 2005.

Darrell Duffie and Henry R Richardson. Mean-variance hedging in continuous time. *The Annals of Applied Probability*, 1(1):1–15, 1991.

P. Dybvig. Distributional analysis of portfolio choice. *Journal of Business*, 61 (3):369–393, 1988a.

P. Dybvig. Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market. *Review of Financial Studies*, 1(1):67–88, 1988b.

Philip H Dybvig. Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market. *Review of Financial Studies*, 1(1):67–88, 1988.

Fang Fang and Cornelis W Oosterlee. A novel pricing method for European options based on fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, 31(2):826–848, 2009.

Hans Föllmer and Alexander Schied. *Stochastic finance: an introduction in discrete time*. Walter de Gruyter, 2011.

Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

Daniel G Goldstein, Eric J Johnson, and William F Sharpe. Choosing outcomes versus choosing products: Consumer-focused retirement investment advice. *Journal of Consumer Research*, 35(3):440–456, 2008.

Christian Gollier. Optimal prevention of unknown risks: A dynamic approach with learning. *Open Access publications from University of Toulouse 1 Capitole*, 2002.

Christian Gollier. Portfolio choices and asset prices: The comparative statics of ambiguity aversion. *Review of Economic Studies*, 78(4):1329–1344, 2011.

Christian Gollier and Miles S Kimball. Toward a systematic approach to the economic effects of uncertainty: characterizing utility functions. Technical report, Discussion paper, University of Michigan, 1996.

Christian Gollier, Bruno Jullien, and Nicolas Treich. Scientific progress and irreversibility: an economic interpretation of the 'precautionary principle'. *Journal of Public Economics*, 75(2):229–253, 2000.

Jun-ya Gotoh and Hiroshi Konno. Third degree stochastic dominance and mean-risk analysis. *Management Science*, 46(2):289–301, 2000.

Richard C Green and Robert A Jarrow. Spanning and completeness in markets with contingent claims. *Journal of Economic Theory*, 41(1):202–210, 1987.

Anne Gundel. Robust utility maximization for complete and incomplete market models. *Finance and Stochastics*, 9(2):151–176, 2005.

Josef Hadar and William R Russell. Rules for ordering uncertain prospects. *The American economic review*, 59(1):25–34, 1969.

Giora Hanoch and Haim Levy. The efficiency analysis of choices involving risk. *Review of Economic Studies*, 36(3):335–346, 1969.

Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

Gero Junike and Konstantin Pankrashkin. Precise option pricing by the cos method - how to choose the truncation interval. *preprint*, 2021.

Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, 2005.

Frank Hyneman Knight. *Risk, uncertainty and profit*, volume 31. Houghton Mifflin, 1921.

Dilip B Madan, Peter P Carr, and Eric C Chang. The variance gamma process and option pricing. *Review of Finance*, 2(1):79–105, 1998.

Mario A Maggi, Umberto Magnani, and Mario Menegatti. On the relationship between absolute prudence and absolute risk aversion. *Decisions in Economics and Finance*, 29(2):155–160, 2006.

Robert C Merton. Optimum consumption and portfolio rules in a continuous-time model. In *Stochastic Optimization Models in Finance*, pages 621–661. Elsevier, 1975.

Jack Meyer. Second degree stochastic dominance with respect to a function. *International Economic Review*, pages 477–487, 1977.

Alfred Müller, Marco Scarsini, Ilia Tsetlin, and Robert L Winkler. Between first-and second-order stochastic dominance. *Management Science*, 63(9): 2933–2947, 2017.

David C Nachman. Spanning and completeness with options. *Review of Financial Studies*, 1(3):311–328, 1988.

Michael Rothschild and Joseph E Stiglitz. Increasing risk: I. a definition. *Journal of Economic theory*, 2(3):225–243, 1970.

Alexander Schied. Optimal investments for robust utility functionals in complete market models. *Mathematics of Operations Research*, 30(3):750–764, 2005.

Moshe Shaked and J George Shanthikumar. *Stochastic orders*. Springer Science & Business Media, 2007.

William F Sharpe, Daniel G Goldstein, and Phil W Blythe. The distribution builder: A tool for inferring investor preferences. *preprint*, 2000.

Bernard Sinclair-Desgagné and H Landis Gabel. Environmental auditing in management systems and public policy. *Journal of Environmental Economics and Management*, 33(3):331–346, 1997.

Hal R Varian. Divergence of opinion in complete markets: A note. *Journal of Finance*, 40(1):309–317, 1985.

Wing-Keung Wong and Chi-Kwong Li. A note on convex stochastic dominance. *Economics Letters*, 62(3):293–300, 1999.